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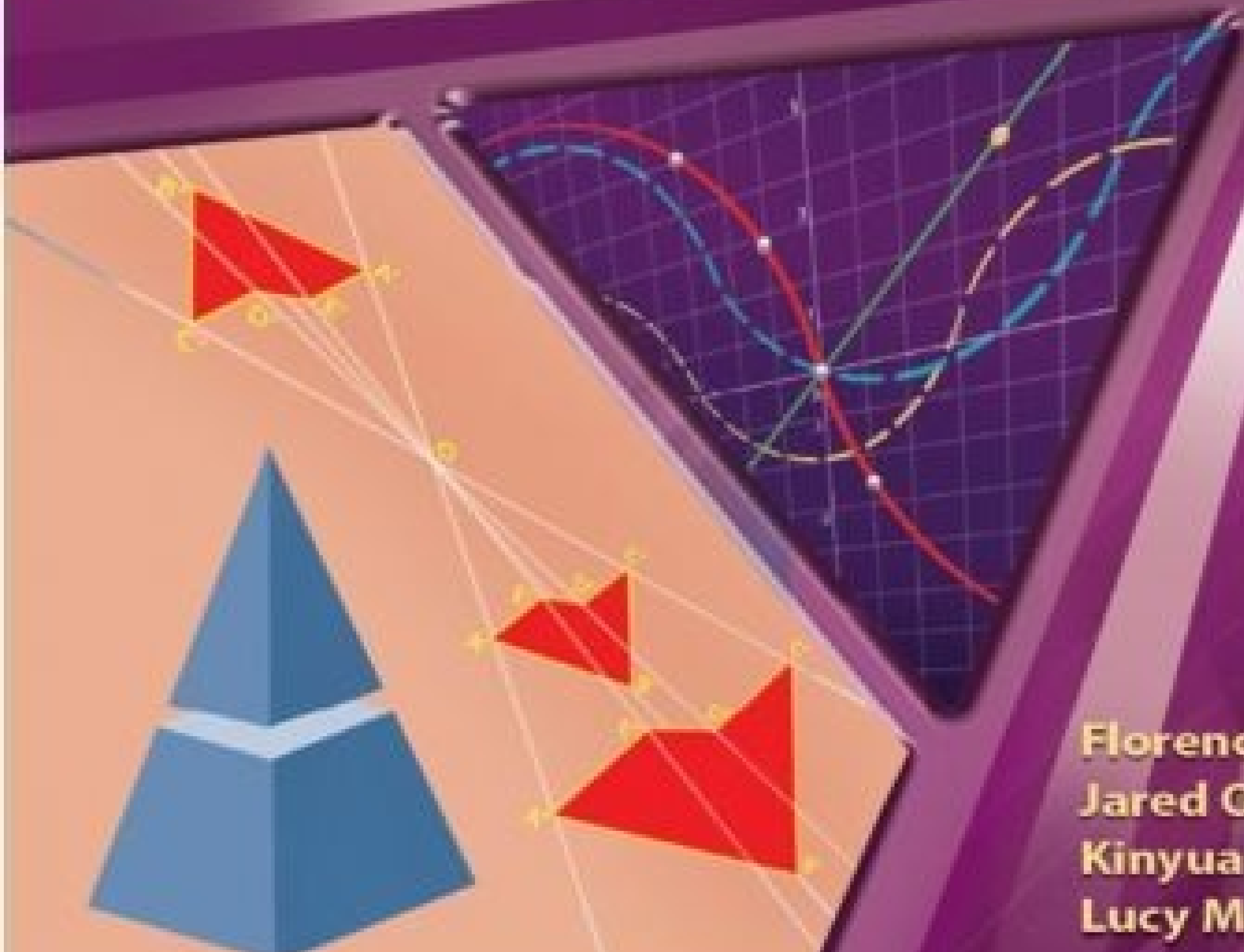
# Junior Secondary Mathematics Form 2



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# JUNIOR SECONDARY

# MATHEMATICS

## FORM TWO

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# Contents

# 1 NUMBER PATTERNS II

In Form 1, we dealt with the general rules which give the terms of sequences. In this chapter, we will generate more sequences, learn how to find the $n$ th term and the number of terms in a sequence. In addition, we will learn how to find the mean of an arithmetic sequence.

## Arithmetic sequences

Examine the following sequences:

(a) 1, 3, 5, 7, 9, …

(b) 2, 4, 8, 16, 32, …

(c) 1, 4, 9, 16, 25, …

In the sequence 1, 3, 5, 7, 9, …, any two consecutive terms differ by 2, i.e.

$3 - 1 = 5 - 3 = 7 - 5 = 9 - 7 = 2$, etc.

Note that the difference in consecutive terms of sequences (b) and (c) is not constant.

A sequence in which any two consecutive terms differ by the same number, i.e. a constant, is called an **arithmetic sequence** . The constant number by which the consecutive terms differ is called the **common difference** .

An arithmetic sequence is generally expressed in the form $a$, $a + d$, $a + 2d$, $a + 3d$, …, where $\boldsymbol{a}$ is the first term and $\boldsymbol{d}$ is the common difference.

Noting that 1st term = $a$, 2nd term = $a + d$, 3rd term = $a + 2d$, 4th term = $a + 3d$, what is the 5th term, 10th term, 15th term and $n$ th term?

In general,

The $n$ th term (denoted as $U_n$ or $T_n$) of an arithmetic sequence is $\boldsymbol{a + (n - 1)d}$ , i.e. $T_n = a + (n - 1)d$ and $d = T_n - T_{n-1}$ .

### *Example 1.1*

*In the sequence* $^-2, 2, 6, 10, \ldots$*, what is the 16th term?*

### *Solution*

*First term,* $a = {}^-2$*, common difference,* $d = 2 - ({}^-2) = 4$*, nth term* $= a + (n - 1)d$

$\therefore$ *16th term* $= {}^-2 + (16 - 1) \times 4 = 58.$

### *Example 1.2*

*The seventh term of an arithmetic sequence is 80. If the first term is 98, what is the 4th term?*

### *Solution*

*Using* $T_n = a + (n - 1)\,d$

$n$ *= 7th term = 80 and* $a = 98$

$80 = 98 + (7 - 1)d$

$^-6d = 98 - 80$

$\therefore d = {}^-3$

*4th term* $= 98 + (4 - 1)\,{}^-3$

$= 98 + 3({}^-3)$

$= 98 - 9 = 89$

## Exercise 1.1

1. For each of the following sequences, find the indicated terms.
   - (a) $1, 3\frac{1}{2}, 6, 8\frac{1}{2}, \ldots$; 20th and 30th terms.
   - (b) $4, 3\frac{1}{2}, 3, 2\frac{1}{2}, \ldots$; 15th and 25th terms.
   - (c) $1, 2\frac{1}{2}, 4, 5\frac{1}{2}, \ldots$; 10th and 20th terms.
   - (d) $1, 0.8, 0.6, \ldots$; 10th and 15th terms.
2. How many terms are in each of the following sequences?
   - (a) $1, 4, 7, \ldots, 61$
   - (b) $4, 4\frac{1}{4}, \ldots, 7$
   - (c) $20, 17\frac{1}{2}, \ldots, {}^-15$
   - (d) $5, 5.9, 6.8, \ldots, 23$

3. The first term of an arithmetic sequence is 3 and the 20th term is $^{-}35$. Find
   (a) the common difference.
   (b) the 10th term.
4. The common difference of an arithmetic sequence is 4 and the 16th term is 65. What is the 20th term?
5. An arithmetic sequence consists of all integers between 1 and 100 which are divisible by 3. How many terms are there?
6. Which term of the arithmetic sequence 4, 13, 22, … is 139?
7. What are the next three terms of the sequence 1, 5, 13, 25, 31, …?
   [Hint: Find the difference between consecutive terms of the sequence. What type of sequence do you get?]

## Mean of an arithmetic sequence

The mean of an arithmetic sequence is the most common average given by,

$$= \frac{\textit{the sum of all the number of terms in a sequence}}{\textit{the total number of terms in the sequence}}$$

***Example 1.3***

*An arithmetic sequence consists of all multiples of 3 between 1 and 50.*

*(a) Find the number of terms in the sequence.*

*(b) What is the mean of the sequence?*

***Solution***

(a) First term, $a = 3$

Common difference, $d = 3$

Last term, $T_n = 48$ (Find the multiple of 3 nearest but less than 50)

$T_n = a + (n - 1)d$

$48 = 3 + (n - 1)3$ (Solve the equation)

$\therefore$ number of terms, $n = 16$

*(b) Add all the terms*

*3 + 6 + 9 + 12 + 15 + 18 + 21 + 24 + 27 + 30 + 33 + 36 + 39 + 42 + 45 + 48 = 408*

$$Mean = \frac{408}{16} = 25.5$$

## Exercise 1.2

1. For each of the following sequences, find

   (a) the number of terms.

   (b) the mean of the sequence.

   (i) $T_n = 32$, 4th term = 16, 5th term = 20

   (ii) $T_n = 55$, a = 1, 2nd term = 7

   (iii) $T_n = 20$, d = 3, 4th term = 11

   (iv) $Tn = \frac{5}{8}$ , a = $\frac{^-7}{8}$ , 2nd term = $\frac{^-3}{8}$

   (v) $T_n = 0.14$, 2nd term = $^-0.1$, 3rd term = $^-0.04$

2. The first three terms are provided for the following sequences. Find

   (a) the other terms.

   (b) the mean of the sequences upto the 7th term.

   (i) $^-2$, 2, 6, ____, ____, ____, ____,

   (ii) $\frac{1}{2}$, $\frac{11}{18}$, $\frac{13}{18}$, ____, ____, ____, ____,

   (iii) 0.9, 0.6, 0.3, ____, ____, ____, ____,

3. Given that the 3rd term of an arithmetic sequence is 90 and the common difference is 9, determine,

   (a) (i) the 2nd term.

   (ii) the 4th term.

   (b) the mean of the first 4 terms in the sequence.

## Arithmetic means of terms

We can also find the arithmetic mean of terms of an arithmetic sequence. Consider the following example.

Let $T_1$ , $T_2$ , $T_3$ , $T_4$ , $T_5$ , ……… be an arithmetic sequence. The common difference is given by $T_2 – T_1$ , or $T_3 – T_2$ , or $T_4 – T_3$ , and so on.

Since the common difference of an arithmetic sequence is constant, then,

$T_2 – T_1 = T_3 – T_2$

$\Rightarrow 2T_2 = T_1 + T_3$

$$\therefore \quad T_2 = \frac{T_1 + T_3}{2}$$

Similarly, $T_3 – T_2 = T_4 – T_3$

$\Rightarrow 2T_3 = T_2 + T_4$

$$\therefore \quad T_3 = \frac{T_2 + T_4}{2}$$

Note that $T_2$ is the mean of $T_1$ and $T_3$ , and $T_3$ is the mean of $T_2$ and $T_4$ .

$T_2$ is referred to as the arithmetic mean of $T_1$ and $T_3$ , and $T_3$ is the arithmetic mean of $T_2$ and $T_3$ .

Thus, when three numbers form three consecutive terms of an arithmetic sequence, the middle number is the **arithmetic mean** of the other two.

Given two terms of an arithmetic sequence, we can generate an arithmetic sequence by generating terms (arithmetic mean) between the given terms using $T_n = a + (n – 1)d$.

***Example 1.4***

*Insert the following arithmetic means between 1 and 9:*

*(a) 1 arithmetic mean*

*(b) 3 arithmetic means*

***Solution***

*(a) Given that there is only one arithmetic mean, then there are only three terms.* $T_1 = a = 1$ *and* $T_3 = 9$

*Using* $Tn = 0 + (n – 1)d$ $(T_n$ *= nth term)*

*then* $9 = 1 + (3 – 1)d$

$2d = 8$

$d = 4$

$\therefore a = 1, T_2 = 1 + 4 = 5, T_3 = 9$

*The sequence is 1, 5, 9.*

(b) *Given that there are 3 arithmetic means, then there are five terms in the sequence.*

$a = 1, T_2, T_3, T_4, T_5 = 9$

Using $T_n = a + (n - 1)d$

then $T_2 = 1 + d$

$T_3 = 1 + (3 - 1)d = 1 + 2d$

$T_4 = 1 + (4 - 1)d = 1 + 3d$

$T_5 = 9 = 1 + 4d$

$\therefore d = 2$

Hence, $T_2 = 3, T_3 = 5, T_4 = 7$

*The sequence is 1, 3, 5, 7, 9.*

## Exercise 1.3

1. Given the terms 2 and 7, insert
   (a) 2 arithmetic means
   (b) 3 arithmetic means
2. Insert the given arithmetic means between 5 and 15:
   (a) 1 arithmetic mean
   (b) 3 arithmetic means
   (c) 5 arithmetic means
3. Given that there are 4 arithmetic means between 6 and 20, what are the next two terms after 20?
4. Insert 6 arithmetic means between 15p and 30p.
5. There are 8 arithmetic means between 25 and 75, what are the next 5 terms of the sequence generated?

## Geometric sequences

Examine the sequences

(a) 2, 4, 8, 16, 32, … and

(b) 4, 9, 16, 25, …

Neither of them is an arithmetic sequence. In sequence (a) , it can be seen that each term is obtained by multiplying the preceding one by 2 and so, if we divide each term by the preceding one, we get a constant value. Thus, we have

$4 \div 2 = 8 \div 4 = 16 \div 8 = 32 \div 16 = 2.$

A sequence in which the ratio between any two consecutive terms is a constant value is called a **geometric sequence** . This constant value is called the **constant ratio** or **common ratio** .

Note that sequence (b) is neither an arithmetic nor a geometric sequence.

In general, the first term is represented by ***a*** and the common ratio is represented by ***r*** . Since in a geometric sequence each term is obtained by multiplying the preceding one by the common ratio, then we have:

1st term = $a$, 2nd term = $ar$, 3rd term = $ar^2$, 4th term = $ar^3$, and so on.

What is the 5th term, 12th term, 15th term, $n$ th term?

We note that:

The $n$ th term ($U_n$ or $T_n$ ) of a geometric sequence is $\boldsymbol{ar^{n-1}}$ , i.e. $T_n = ar^{n-1}$ .

In general, a geometric sequence is in the form

$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$

## *Example 1.5*

*What are the 3rd, 4th and nth terms in the sequence $^-8, 24, \ldots$?*

## *Solution*

*This is a geometric sequence with the first term $a = {}^-8$ and common ratio $r = \frac{24}{^-8} = {}^-3$.*

*Using* $T_n = ar^{n-1}$

$\therefore$ *3rd term* $= {}^-8 \times ({}^-3)^{3-1} = {}^-8({}^-3)^2$

$= {}^-8 \times ({}^-3 \times {}^-3)$

$= {}^-8 \times 9 = {}^-72$

*4th term* $= {}^-8 \times ({}^-3)^{4-1} = {}^-8({}^-3)^3$

$= {}^-8 \times ({}^-3 \times {}^-3 \times {}^-3)$

$= {}^-8 \times {}^-27 = 216$

*nth term* $= 8 \times ({}^-3)^{n-1}$

### *Example 1.6*

*The first term of a geometric sequence is 12 and the 4th term is –. Write down the first 4 terms of the sequence.*

### *Solution*

*4th term* $= 12 \times (r^{4-1}) = \frac{3}{2}$

$= 12r^3 = \frac{3}{2}$

$r^3 = \frac{1}{8} = 0.125$

$r = \sqrt[3]{0.125} = 0.5$

*2nd term* $= 12 \times (0.5)^1 = 6$

*3rd term* $= 12 \times (0.5)^2 = 3$

*The sequence is 12, 6, 3,* $\frac{3}{2}$ *.*

## Exercise 1.4

1. For each of the following sequences find the common ratio and the next term.

   (a) 1, 3, 9, …

   (b) 1, $\frac{2}{3}$, $\frac{4}{9}$, …

   (c) 1, 0.8, 0.64, …

   (d) 2, $^-4$, 8, …;

   (e) $\frac{2\,401}{16}$, $\frac{343}{8}$, $\frac{49}{4}$, …

   (f) 6, $^-0.6$, 0.06, …

2. 12, *b*, 75 are consecutive terms of a geometric sequence. What is the

common ratio for the sequence?

3. The first term in a geometric sequence is 512. The fourth term is 1. Find the first 4 terms of the sequence.
4. The first term of a geometric sequence is 1.1 and the fourth term is 1.464 1. What is the common ratio for the sequence?
5. The common ratio of a geometric sequence is $^{-}2$ and the second term is $^{-}1$. What is the first term?

# 2 PYTHAGORAS' THEOREM

## The theorem

Given the lengths of two sides of a right-angled triangle, how would you find the length of the third side?

For a very long time, it has been known, as a matter of fact, that a triangle with sides 3, 4 and 5 units is right-angled. However, this is only one isolated case and does not provide an answer to the question above. The following activities will help us establish the relationship between the lengths of the sides and hence answer the question.

### *Activity 2.1*

Consider a floor that is tiled with tiles of the same size, each is a right-angled isosceles triangle, as in Fig. 2.1 .

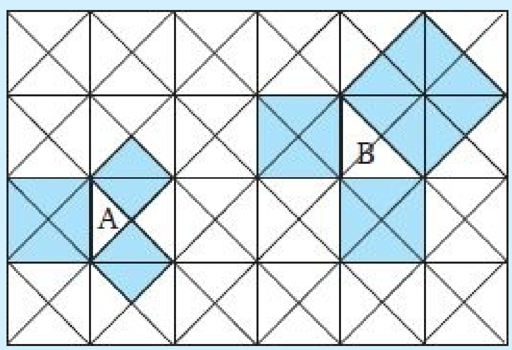



*Fig. 2.1*

1. Make a copy of this figure on a larger scale.
2. Pick any of the small triangular tiles (e.g. A) and shade the three squares standing on its sides. The squares on the shorter sides are equal (each comprising of two triangular tiles). How many tiles make the square on the longest side (called the **hypotenuse** )?
3. Shade a triangle composed of two triangular tiles (e.g. B). This triangle is similar to the previous one. Count the number of tiles that make the squares on its sides.
4. Continue shading triangles of the same shape but of increasing size and record your results as in Table 2.1 .

| No. of tiles making the triangle | No. of tiles making the square on | | |
|---|---|---|---|
| | 1st short side $a$ | 2nd short side $b$ | Longest side $c$ |
| 1 | 2 | 2 | 4 |
| 2 | | | |
| 3 | | | |
| | | | |
| | | | |
| | | | |

*Table 2.1*

What is the relationship between the values *a* , *b* and *c* ? Is this fact true if the right-angled triangle is not isosceles? Verify this by carrying out Activities 2.2 and 2.3.

## *Activity 2.2*

1. Draw ΔABC, right-angled at B on a stiff paper, such that BC is longer than AB. Construct the three squares on its sides, as in Fig. 2.2 .

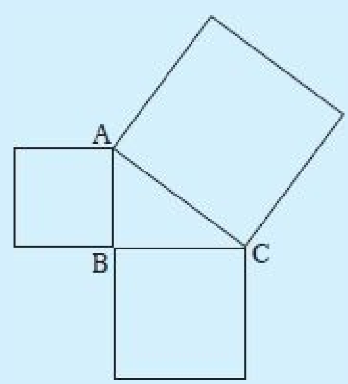



*Fig. 2.2*

2. Locate the centre P of the square on side BC. Through P, construct a line perpendicular to AC and another line parallel to AC, to subdivide the square into four pieces as in Fig. 2.3 .

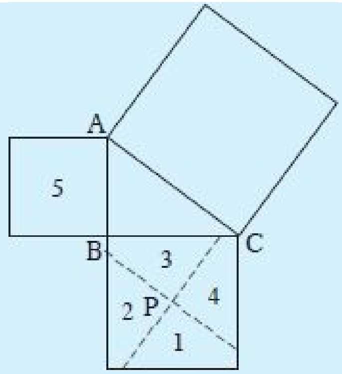



*Fig. 2.3*

3. Cut out the squares on AB and BC. Cut up the square on BC into the four pieces indicated. Arrange the pieces to cover completely the square on AC (Fig. 2.4 ). Note that the pieces can be moved into their new positions without rotating any of them or turning them over. (This is like a jigsaw puzzle and is referred to as **Perigal's dissection** ).

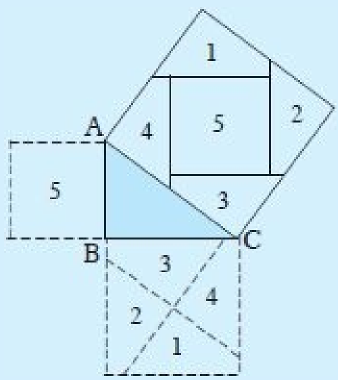



*Fig. 2.4*

What can you say about the areas of the squares on the two shorter sides of the triangle?

**Point of interest**

*Henry Perigal* (1801–1898), an Englishman of French descent, was a stockbroker and amateur astronomer and geometer. He spent a great part of his life studying geometric theorems and problems by dissections and transpositions. In 1830, he announced a single and elegant proof of the theorem of Pythagoras, which has subsequently been referred to as Perigal's dissection.

Though Perigal arrived at it independently, this type of dissection to prove the $3^2 + 4^2 = 5^2$ relationship of the sides of a right-angled triangle had been known by other mathematicians much earlier than him. For example, the Chinese had a similar proof dating back to 300 BC. However, Perigal was probably the first person to extensively write and publish on this proof and hence the use of his name.

### *Activity 2.3*

1. Draw any right-angled triangle ABC, with sides labelled as in Fig. 2.5 .

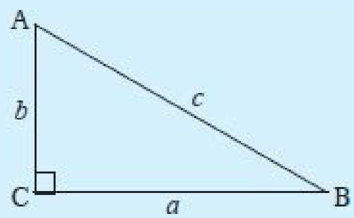



*Fig. 2.5*

2. Construct a square on each side of the triangle.
3. Measure the length of the side of each square and calculate the area of the square.
4. State the relationship between the areas of the three squares.

By now, you should have established what is known as the **Pythagorean relationship** or **Pythagoras' theorem** , stated as follows:

In a right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides, i.e.

$c^2 = a^2 + b^2$ or $a^2 + b^2 = c^2$

**Point of interest**

Pythagoras' theorem is named after a great Greek mathematician, Pythagoras of Samos, who lived from about 580 BC to 500 BC. Pythagoras travelled a great deal in Egypt before finally settling down in a Greek colony in the south of Italy.

Pythagoras was both a philosopher and a mathematician (a geometer). He was a leader of a religious brotherhood whose members believed in the transmigration of souls, i.e. that the spirit of man or beast moved on after death to another man or beast. They also believed in strict taboos for self discipline (e.g. they would not eat beans) and in the supreme importance of numbers in the creation of the universe. They sought to interpret all things through numbers.

Pythagoras learned many facts about mathematics from the Egyptians. One of the facts which certainly impressed him was that a triangle of sides 3, 4 and 5 units is right-angled, a fact that was known long before his time.

However, the truth of the relationship $a^2 + b^2 = c^2$, about *any* right-angled triangle, was not known.

It was probably in furthering religious ideas that Pythagoras, or one of his disciples, discovered the theorem.

## Proof of Pythagoras' theorem

### *Activity 2.4*

For Pythagoras' theorem to be completely general, think of a triangle T whose sides are of lengths $a$, $b$ and $c$ units, as in Fig. 2.6 .

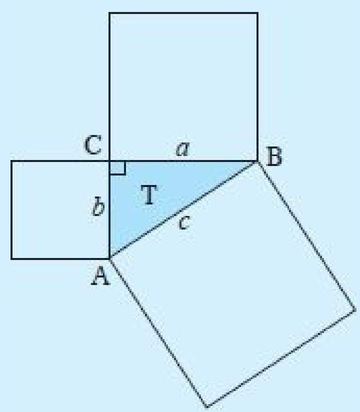



*Fig. 2.6*

1. Rotate triangle T through 90° in a clockwise direction about the centre O of the square on the longest side. Triangle T is mapped onto triangle T′ as in Fig. 2.7 .

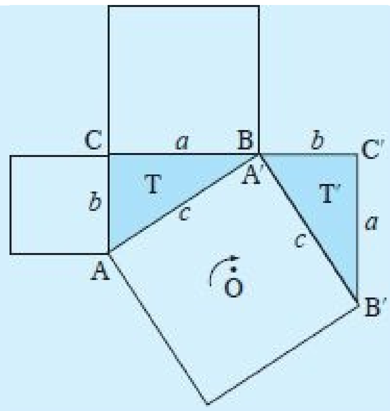



*Fig. 2.7*

Note that line CBC′ is straight and C′B′ is at right angles to it.

2. Rotate triangle T′ in a clockwise direction about O twice through 90° in each case, to positions T′′ and T′′′ as in Fig. 2.8 .

   Note that the figure CC′C′′C′′′ that is obtained after the three rotations, is a square of side ($a$ + $b$ ) units.

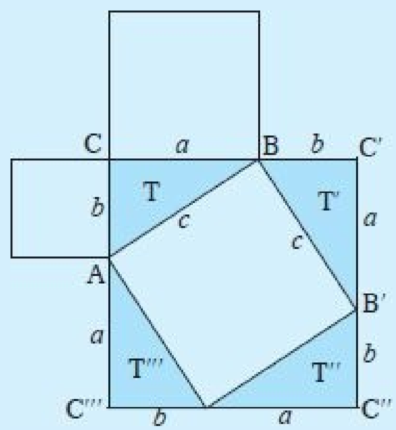



*Fig. 2.8*

3. Look at this square as follows:

   (a) As the central square (of a side $c$ ) plus four equal triangles of sides $a$ and $b$ units. Thus, the area of the square is

   $c^2 + 4 \times \frac{1}{2} \times a \times b$

   = ($c^2$ + 2$ab$ ) square units ……… (i)

   (b) As a square of side ($a$ + $b$ ) units, whose area is $(a + b)^2$

   = ($a^2 + b^2$ + 2$ab$ ) square units ……(ii)

   The two expressions, (i) and (ii), are for the same area, and so,

$c^2 + 2ab = a^2 + b^2 + 2ab$

i.e. $c^2 = a^2 + b^2$

## Using Pythagoras' theorem

As we have seen, Pythagoras' theorem concerns areas. Its main use, however, is in calculating lengths. It also provides us with a test for a right-angled triangle, namely:

A triangle is right-angled whenever the square of the length of the longest side equals the sum of the squares of the lengths of the other two sides.

### *Example 2.1*

*Calculate the length of the third side of the triangle in Fig. 2.9 .*

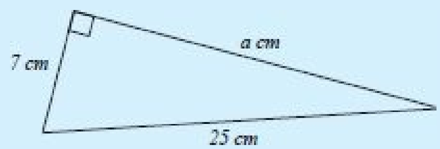



*Fig. 2.9*

### *Solution*

*Using Pythagoras' theorem,*

$$25^2 = a^2 + 7^2$$

i.e. $625 = a^2 + 49$

$$\therefore a^2 = 625 - 49$$

$$a^2 = 576$$

$$\Rightarrow a = \sqrt{576} = 24 \text{ cm}$$

*i.e. the length of the third side is 24 cm.*

### *Example 2.2*

*Find the length of AB in Fig. 2.10 .*

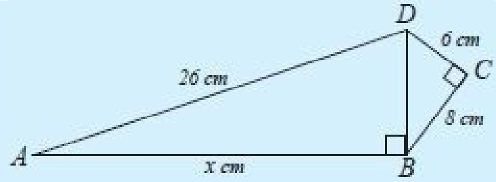



*Fig. 2.10*

### *Solution*

*ΔBCD is a right-angled triangle*

$\therefore BD^2 = 8^2 + 6^2$ *(by Pythagoras' theorem)*

$= 100 \text{ cm}^2$

*ΔABD is right-angled*

$\therefore 26^2 = x^2 + BD^2$ *(by Pythagoras' theorem)*

*i.e.* $26^2 = x^2 + 100$

$\therefore x^2 = 676 - 100$

$x^2 = 576$

$\Rightarrow \quad x = \sqrt{576} = 24$

$\therefore AB = 24$ *cm*

### *Example 2.3*

*Find out whether a triangle with sides 11, 15 and 18 cm is right-angled.*

### *Solution*

*The two shorter sides are 11 cm and 15 cm in length. The sum of the squares of their lengths is*

$11^2 + 15^2 = 121 + 225$

$= 346$

*The square of the length of the hypotenuse is*

$18^2 = 324$

*Now* $11^2 + 15^2 \neq 18^2$

$\therefore$ *the triangle is not right-angled.*

## Exercise 2.1

1. The triangle in Fig. 2.11 is right-angled where shown. A, B and C are

the areas of the squares. Find the third area in each of the following cases.

(a) A = 28 $cm^2$ , B = 17 $cm^2$
(b) B = 167 $cm^2$ , C = 225 $cm^2$
(c) A = 4.53 $m^2$ , C = 6.89 $m^2$

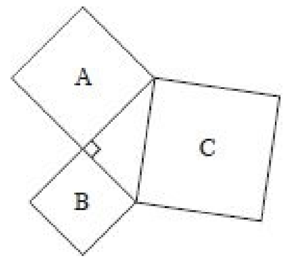



*Fig. 2.11*

2. Fig. 2.12 shows a right-angled triangle, with all measurements in centimetres.

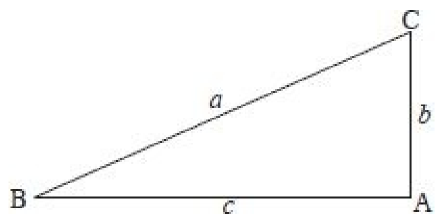



*Fig. 2.12*

(a) If $b$ = 6 and $c$ = 8, find $a$ .
(b) If $b$ = 8 and $c$ = 15, find $a$ .
(c) If $b$ = 9 and $a$ = 15, find $c$ .
(d) If $a$ = 50 and $c$ = 48, find $b$ .

3. In triangle ABC, AB = 3 cm, BC = 5 cm and ∠ABC = 90°. Find AC.

4. In ΔLMN, LM = 4 cm, LN = 6 cm and ∠LMN = 90°. Find MN.

5. Which of the following measurements would give a right-angled triangle?

(a) 6 cm by 8 cm by 10 cm
(b) 5 cm by 12 cm by 13 cm
(c) 4 cm by 16 cm by 17 cm
(d) $7\frac{1}{2}$ cm by 10 cm by $12\frac{1}{2}$ cm
(e) 9 cm by 30 cm by 35 cm

(f) 12 cm by 35 cm by 37 cm
(g) 12 m by 60 m by 61 m
(h) 21 m by 90 m by 101 m
(i) 20 m by 21 m by 28 m
(j) 28 m by 45 m by 53 m
(k) 27 m by 35 m by 50 m
(l) 33 m by 44 m by 55 m
(m) 4 m by $7\frac{1}{2}$ m by $8\frac{1}{2}$ m
(n) 14 m by 48 m by 50 m
(o) 2.7 m by 36.4 m by 36.5 m
(p) 2.9 m by 42 m by 42.1 m

6. In Fig. 2.13 , all the measurements are in centimetres. Find the lengths marked by letters.

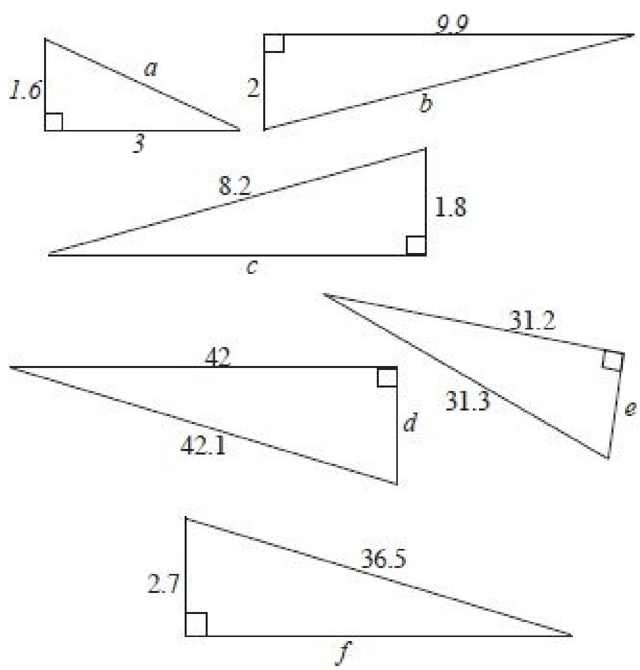



*Fig. 2.13*

7. In Fig. 2.14 , all the measurements are in metres. Find the lengths marked by letters.

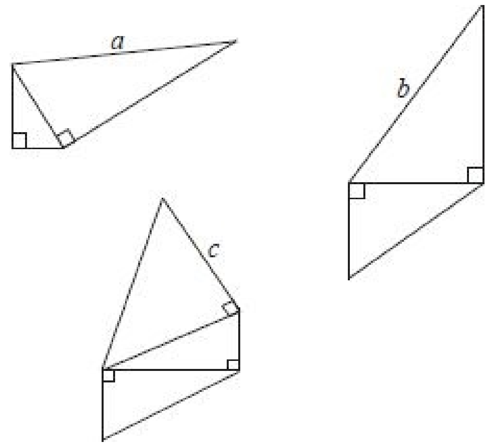



*Fig. 2.14*

8. The sides of a rectangle are 7.8 cm and 6.4 cm long. Find the length of the diagonal of the rectangle.
9. The length of the diagonal of a rectangle is 23.7 cm and the length of one side is 18.8 cm. Find its perimeter.

## Pythagorean triples

Revisit your answers to Question 5 of Exercise 2.1. You notice that some of the values, like (6, 8, 10), give right-angled triangles while values such as (4, 16, 17) do not. Such sets of values which give right-angled triangles are known as **Pythagorean triples** or **Pythagorean numbers** .

A **Pythagorean triple** *(a , b, c,)* is a group of three numbers which give the respective lengths of the sides and the hypotenuse of a right-angled triangle and are related thus:

$$a^2 + b^2 = c^2$$

The group (3, 4, 5) is the most famous and most commonly used Pythagorean triple. The triple was known even before the time of Pythagoras. It was, and is still, used for setting out the base lines on tennis courts and other sports pitches.

There are many other Pythagorean triples. Now complete the following patterns to discover some more.

(a) (3, 4, 5) $\rightarrow 3^2 = 4 + 5$

$(5, 12, 13) \rightarrow 5^2 = 12 + 13$

$(7, 24, 25) \rightarrow 7^2 = 24 + 25$

$(9, 40, 41) \rightarrow 9^2 = 40 + 41$

$(11, \ldots, \ldots) \rightarrow 11^2 =$

$(13, \ldots, \ldots) \rightarrow 13^2 =$

$(15, \ldots, \ldots) \rightarrow 15^2 =$

$(17, \ldots, \ldots) \rightarrow 17^2 =$

$(19, \ldots, \ldots) \rightarrow 19^2 =$

$(21, \ldots, \ldots) \rightarrow 21^2 =$

(b) $(6, 8, 10) \rightarrow \frac{1}{2}$ of $6^2 = 8 + 10$

$(8, 15, 17) \rightarrow \frac{1}{2}$ of $8^2 = 15 + 17$

$(10, 24, 26) \rightarrow \frac{1}{2}$ of $10^2 = 24 + 26$

$(12, 35, 37) \rightarrow \frac{1}{2}$ of $12^2 = 35 + 37$

$(14, \ldots, \ldots) \rightarrow \frac{1}{2}$ of $14^2 =$

$(16, \ldots, \ldots) \rightarrow \frac{1}{2}$ of $16^2 =$

$(18, \ldots, \ldots) \rightarrow \frac{1}{2}$ of $18^2 =$

$(20, \ldots, \ldots) \rightarrow \frac{1}{2}$ of $20^2 =$

$(22, \ldots, \ldots) \rightarrow \frac{1}{2}$ of $22^2 =$

$(24, \ldots, \ldots) \rightarrow \frac{1}{2}$ of $24^2 =$

These two patterns provide ways of finding Pythagorean triples. But they do not seem to give all the possible sets of such triples. Is there a general way of finding Pythagorean triples?

Now consider the following.

(a) $2^2 - 1^2$ , $2 \times 2 \times 1$, $2^2 + 1^2$

(b) $3^2 - 1^2$ , $2 \times 3 \times 1$, $3^2 + 1^2$

(c) $4^2 - 2^2$ , $2 \times 4 \times 2$, $4^2 + 2^2$

(d) $5^2 - 3^2$ , $2 \times 5 \times 3$, $5^2 + 3^2$

Are these all Pythagorean triples?

It is a fact that:

Given any two positive integers $m$ and $n$, where $m > n$, we always obtain the Pythagorean triple:

$$(m^2 - n^2, 2mn, m^2 + n^2)$$

## Exercise 2.2

1. (5, 12, 13) is a Pythagorean triple.
   (a) Write down four multiples of it.
   (b) Are all the four multiples in (a) Pythagorean triples?
   (c) Using a multiplier $n$ and any Pythagorean triple ($a$ , $b$ , $c$ ,) state the general result for such multiples as in (a) .
2. Find out if the following are Pythagorean triples.
   (a) (7, 24, 25)
   (b) (8, 15, 17)
   (c) (15, 22, 27)
   (d) (28, 43, 53)
   (e) (11, 60, 61)
   (f) (20, 21, 29)
3. The following are the dimensions of two triangles. Which one of them is a right-angled triangle?
   (a) 15 cm, 30 cm, 35 cm
   (b) 33 cm, 56 cm, 65 cm
4. Use the following numbers to generate Pythagorean triples.
   (a) 1 and 4
   (b) 1 and 5
   (c) 6 and 2
   (d) 3 and 8
5. Complete the following Pythagorean triples.
   (a) (25, …, …)
   (b) (31, …, …)
   (c) (43, …, …)
   (d) (49, …, …)
   (e) (30, …, …)
   (f) (38, …, …)
   (g) (44, …, …)

(h) (64, …, …)

## Using Pythagoras' theorem in real life situations

As we have seen, Pythagoras' theorem connects the areas of actual squares. Its main use, however, is in calculating lengths without having to draw any squares. The theorem also acts as a test for right-angled triangles.

There are many real life situations which require the use of Pythagoras' theorem.

### *Example 2.4*

*A ladder, 3.9 m long, leans against a wall. If its foot is 1.2 m from the wall, how high up the wall does it reach?*

### *Solution*

*Fig. 2.15 is an illustration of the situation. Note that the ground must be assumed to be horizontal and level and hence at right angles to the wall.*

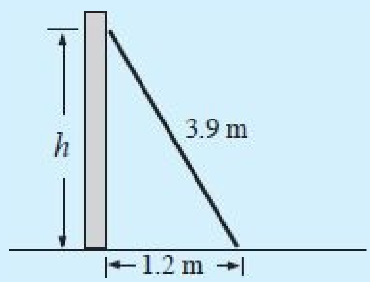



*Fig. 2.15*

*By Pythagoras' theorem,*

$$h^2 + 1.2^2 = 3.9^2$$

$$\Rightarrow h^2 = 3.9^2 - 1.2^2$$

$$\therefore h = \sqrt{3.9^2 - 1.2^2}$$

$$h = 3.71 \text{ m (2 d.p.)}$$

*Thus the ladder reaches 3.71 m up the wall.*

## Exercise 2.3

1. Fig. 2.16 shows a television antenna. Find the length of the wire AB holding the antenna.

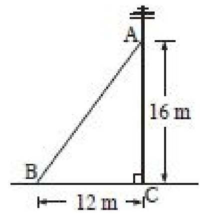



*Fig. 2.16*

2. A ladder reaches the top of a wall of height 6 m when the end on the ground is 2.5 m from the wall. What is the length of the ladder?
3. The length of a diagonal of a rectangular flower bed is 24.6 m and the length of one side is 18.9 m. Find the perimeter and area of the flower bed.
4. A piece of rope with 12 knots that are equally spaced has been laid out and pinned down on the ground as in Fig. 2.17 .

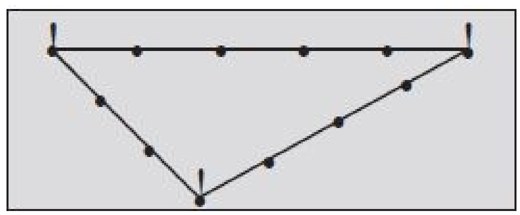


*Fig. 2.17*

   (a) What can you say about the triangle whose corners the stakes mark?
   (b) Does it matter how great the distance between the knots is?

5. The chalkboard in your classroom is rectangular and it measures 2.2 m by 1.2 m. What is the length of the longest straight line that can be drawn on it?
6. Fig. 2.18 shows a road that turns through a right angle to go round a rectangular recreational garden in your town. To save time, people on foot cut off the corner, thus making a path that meets the road at 45°. If the path is 48 m long, find the distance that the people save?

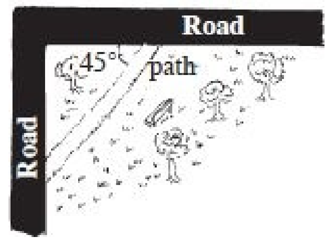



*Fig. 2.18*

7. A hall is 16 m long, 14 m wide and 9 m high. Find the length of the diagonal of the floor.

8. Fig. 2.19 represents a roof truss which is symmetrical about QS. Beam PQ is 5 m long, strut TS 2.4 m long and the distance TQ is 1.8 m.
   (a) Find the height QS.
   (b) Hence, find the span PR of the roof.

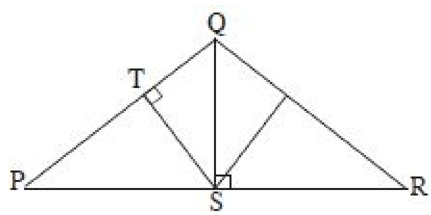



*Fig. 2.19*

# 3 ALGEBRAIC PROCESSES II

## Simple algebraic expressions

In Form 1, we learnt how to expand expressions of the form $a(x + y)$. In this case, each term inside the bracket is multiplied by the number outside the bracket.

**Remember:**
If the number outside the bracket is negative, the sign of each term in the bracket changes when the bracket is expanded or opened.

### *Example 3.1*

*Expand*

(a) $3(4x - 5)$

(b) $^-4(2x + 7)$

(c) $(2x + 3)(^-x)$

### *Solution*

(a) *Note that* $4x - 5 = 4x + {^-5}$, *so that*

$3(4x - 5) = 3 \times 4x + 3 \times {^-5}$ *(Multiplication is distributive over addition)*

$= 12x + {^-15}$

$= 12x - 15$

(b) $^-4(2x + 7) = {^-4} \times 2x + {^-4} \times 7$

$= {^-8x} + {^-28}$

$= {^-8x} - 28$

*(Note the sign change from + to –)*

*(c)* $(2x - 3)(^{-}x) = {}^{-}x \times 2x + (^{-}x) \times (^{-}3)$

$= {}^{-}2x^2 + 3x$

*(Note the sign change from + to – and from – to +)*

## Exercise 3.1

1. Expand the following.

   (a) $3(x + 4)$
   (b) $4(x - 6)$
   (c) $4(7x - 3)$
   (d) $^{-}2(8a - 5)$
   (e) $3(4 - 3a)$
   (f) $3(4a + 5b - c)$
   (g) $^{-}5(x + 5)$
   (h) $5(6 - 2y)$
   (i) $(2 - x)x$
   (j) $x(x - 1)$
   (k) $a(a + 7)$
   (l) $b(b - 4)$
   (m) $(4b - 1)(-b)$
   (n) $^{-}x(2x - 3)$
   (o) $^{-}x(4x - 5)$
   (p) $x(y + z)$
   (q) $(4x - 3)xy$
   (r) $^{-}2x(^{-}3x - 5)$

2. Expand and simplify:

   (a) $2(x + 1) + 3(x + 2)$
   (b) $4(x - 5) + 3(2x + 1)$
   (c) $3(3y - 5) - 7(2y + 3)$
   (d) $y(y + 4) + 3(y + 4)$
   (e) $z(z + 1) + 1(z + 1)$
   (f) $^{-}x(x + 5) + 5(x - 5)$
   (g) $y(y - 6) - 6(y - 6)$
   (h) $5(2x - 5) + 8(3x - 1)$
   (i) $3(4 - 5x) - 2(5 - 4x)$

(j) $y(y + 2) - 3y(y + 2)$
(k) $t(t - 5) - 5(t - 5)$

## Binomial expansions

An algebraic expression consisting of two terms is called a **binomial expression** . Thus, $3x + 5$, $a + b$ , $2x - y$ are examples of binomial expressions.

How do we find the product of two binomial expressions?

Recall: $p(x + y) = px + py$

Likewise, $(a + b)(x + y) = (a + b)x + (a + b)y$

[$(a + b)$ taken as a single quantity]

$= ax + bx + ay + by$ .

Thus, each term in one bracket must be multiplied by each term in the other bracket and the results added.

i.e. $(a + b)(x + y) = ax + ay + bx + by$

### *Example 3.2*

*Multiply and simplify*

(a) $(x + 2)(x + 3)$
(b) $(3x - 2)(2x - 3)$

### *Solution*

(a) $(x + 2)(x + 3) = x \times (x + 3) + 2 \times (x + 3)$

$= x \times x + x \times 3 + 2 \times x + 2 \times 3$

*(Each term in first bracket multiplied by each term in second bracket)*

$= x^2 + \underbrace{3x + 2x}_{\text{like terms}} + 6$

$= x^2 + 5x + 6$ *(Combine like terms)*

(b) $(3x - 2)(2x - 3) = 3x(2x - 3) - 2(2x - 3)$

$= 6x^2 \underbrace{- 9x - 4x}_{\text{like terms}} + 6$

$= 6x^2 - 13x + 6$

## Exercise 3.2

Find the products of the following binomial expressions and simplify where possible.

1. (a) $(x + 2)(x + 5)$
   (b) $(x + 5)(x + 1)$
   (c) $(x - 2)(x - 3)$
   (d) $(a - b)(a - b)$
   (e) $(x + 4)(x - 4)$
   (f) $(x + 3)(x + 3)$
   (g) $(x + y)(c + d)$
   (h) $(a + b)(a + b)$

2. (a) $(x + 3)(2x + 1)$
   (b) $(2x + 3)(3x + 2)$
   (c) $(x + 7)(4x + 1)$
   (d) $(3a + 2)(a + 4)$
   (e) $(y - 4)(5y - 3)$
   (f) $(4a - 3)(3a - 4)$
   (g) $(7 + 4x)(x + 2)$
   (h) $(1 + x)(9 + 5x)$

3. (a) $(l + m)(l + n)$
   (b) $(p + q)(r + 4)$
   (c) $(a + b)(c + z)$
   (d) $(m + 3)(m + n)$
   (e) $(a + b)(c - d)$
   (f) $(3x - 2y)(x + 2)$

(g) $(p - q)(3p + 2q)$
(h) $(x + y)(3x - 6y)$

## Binomial products (quadratic identities)

Three special binomial products appear so often in algebra that their expansions can be stated with minimum computation.

## Binomial squares

In arithmetic, we know that $2^2$ means $2 \times 2 = 4$, 3 means $3 \times 3 = 9$, and so on.

In algebra, $(a + b)^2$ means $(a + b) \times (a + b)$.

Thus, $(a + b)^2 = (a + b)(a + b)$

$$= a(a + b) + b(a + b)$$

$$= a^2 + ab + ba + b^2$$

$$= a^2 + 2ab + b^2 \text{ (since } ab = ba\text{)}$$

Also $(a - b)^2$ means $(a - b) \times (a - b)$

Thus, $(a - b)^2 = (a - b)(a - b)$

$$= a(a - b) - b(a - b)$$

$$= a^2 - ab - ba + b^2$$

$$= a^2 - 2ab + b^2$$

$(a + b)^2$ and $(a - b)^2$ are called **squares of binomials** or simply **perfect squares**
.

The three terms of the product can be obtained through the following procedure.

1. The first term of the product is the square of the first term of the binomial, i.e. $(a)^2 = a^2$ .
2. The second term of the product is two times the product of the two terms of the binomial, i.e. $2 \times (a \times b) = 2ab$
3. The third term of the product is equal to the square of the second term of the binomial, i.e. $(b)^2 = b^2$ .

Thus,

$\boldsymbol{(a + b)^2 = a^2 + 2ab + b^2}$ and **not** $a^2 + b^2$ .

This is a common error which must be avoided.

Similarly,

$\boldsymbol{(a - b)^2 = a^2 - 2ab + b^2}$ and **not** $a^2 - b^2$ .

The **square of a binomial** always gives a **trinomial** , (i.e. an expression having three terms), also known as a **quadratic expression** .

## A difference of two squares

A third special product comes from multiplying the sum and difference of the same two terms.

Consider the product $(a + b)(a - b)$.

$$
\begin{aligned}
(a + b)(a - b) &= a(a - b) + b(a - b) \\
&= a^2 - ab + ab - b^2 \\
&= a^2 - b^2
\end{aligned}
$$

(Since $ab = ba$, then $^{-}ab + ba = {}^{-}ab + ab = 0$)

This product may be obtained by:

1. Squaring the first term of the factors.
2. Subtracting the square of the second term of the factors.

The result $\boldsymbol{(a + b)(a - b) = a^2 - b^2}$ is called a **difference of two squares** .

The expansions

$(a + b)^2 = a^2 + 2ab + b^2$

$(a - b)^2 = a^2 - 2ab + b^2$ , and

$(a + b)(a - b) = a^2 - b^2$

are known as **quadratic identities** .

## Use of area to derive the quadratic identities

In this section, we use the idea of area of a rectangle to derive the three identities. It is hoped that this section will help you appreciate that when expanding the algebraic expressions, we are looking for areas of some rectangles (and squares).

Fig. 3.1 (a) is a square ABCD with sides of length ($a + b$ ).

Hence area of ABCD = $(a + b)^2$ ………(1)

Fig. 3.1 (b) is the same square ABCD [Fig. 3.1 (a)]. In it is a small square AEFG of lengths $a$ . The square ABCD [Fig. 3.1 (b)] can be divided as shown in Fig. 3.1 (c).

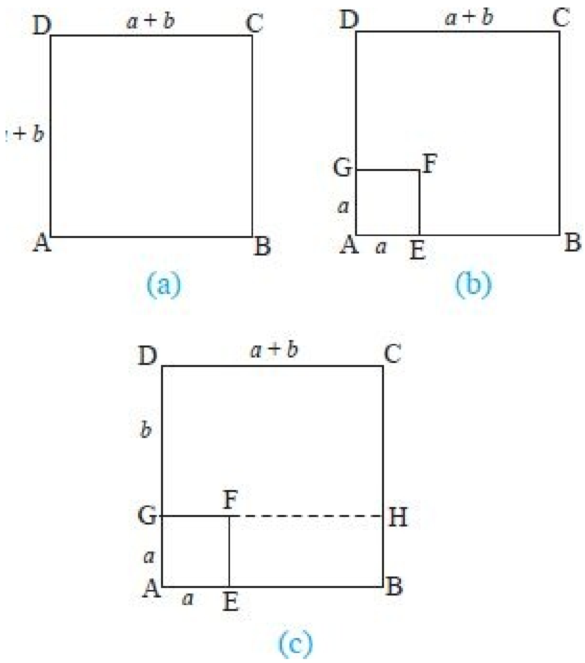



*Fig. 3.1*

Hence area of ABCD = Area of AEFG + Area of EBHF + Area of GHCD

Area of AEFG = $a^2$

Area of EBHF = $ab$

Area of GHCD = $b(a + b) = ab + b^2$

Thus area of ABCD = $a^2 + ab + ab + b^2$

$= a^2 + 2ab + b^2$ ………(2)

Since area of ABCD = $(a + b)^2$ (from (1))

Then

$$(a + b)^2 = a^2 + 2ab + b^2$$

Fig. 3.2 (a) is a square ABCD with sides of length $a$ .

Fig. 3.2 (b) shows the same square ABCD of length $a$ .

(a)

(b)

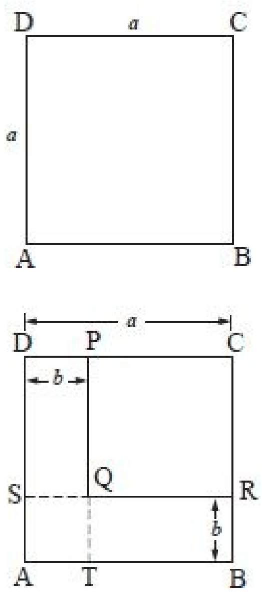



*Fig. 3.2*

PQRC is a square contained in ABCD such that DP = SQ = AT = SA = QT = RB = $b$

This means PQRC is a square of length $(a – b)$.

Area of PQRC = $(a – b)^2$ .........(1)

But area of PQRC = Area of ABCD – (Area of DPQS + Area of SQTA + Area of QTBR)

Area of ABCD = $a^2$

Area of DPQS = $b(a – b) = ab – b^2$

Area of SQTA = $b^2$

Area of QTBR = $b(a – b) = ab – b^2$

∴ Area of PQRC = $a^2 – [ab – b^2 + b^2 + ab – b^2]$

$= a^2 – (2ab – b^2)$

$= a – 2ab + b^2$ .........(2)

Hence from (1) and (2)

$$(a – b)^2 = a^2 – 2ab + b^2$$

Fig. 3.3 (a) is a rectangle ABCD with sides of length $(a + b)$ and $(a - b)$.

Hence area of ABCD = $(a + b)(a - b)$ ………(1)

Fig. 3.3 (b) is the same rectangle ABCD in Fig. 3.3 (a), with PB = $b$ hence AP = $a$.

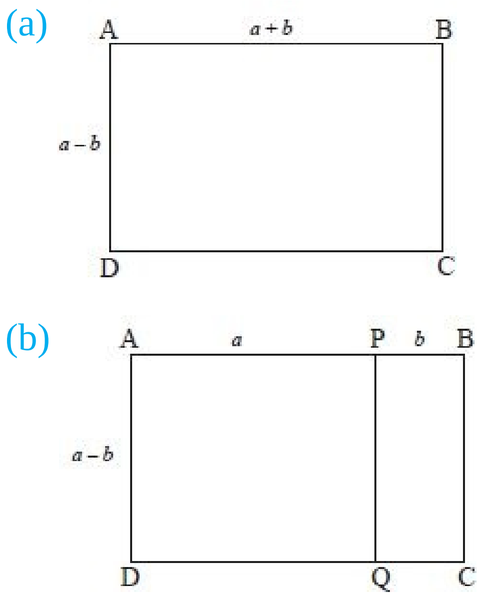



*Fig. 3.3*

Area of APQD = $a(a - b) = a^2 - ab$

Area of PBCQ = $b(a - b) = ab - b^2$

Area of ABCD = Area of APQD + Area of PBCQ

$$= a^2 - ab + ab - b^2$$

$$= a^2 - b^2 \quad \text{………(2)}$$

Comparing (1) and (2) we get

$$(a + b)(a - b) = a^2 - b^2$$

We have seen that given squares of sides $(a + b)$ and $(a - b)$ and rectangle of sides $(a + b)$ and $(a - b)$, their areas are given by

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

These are known as the three **quadratic identities** .

## Applying the difference of two squares

### *Example 3.3*

*Find (a) $12^2$ (b) $18^2$ (c) $102 \times 98$*

### *Solution*

(a) $12 = 10 + 2$

$$\therefore 12^2 = (10 + 2)^2 = 10^2 + 2 \times 10 \times 2 + 2^2$$
$$= 100 + 40 + 4$$
$$= 144$$

(b) $18 = 20 - 2$

$$\therefore 18^2 = (20 - 2)^2 = 20^2 - 2 \times 20 \times 2 + 2^2$$
$$= 400 - 80 + 4$$
$$= 324$$

(c) $102 \times 98 = (100 + 2)(100 - 2)$

$$= 100^2 - 2^2$$
$$= 10\,000 - 4$$
$$= 9\,996$$

## Exercise 3.3

1. Use the quadratic identities to calculate the following.

   (a) $11^2$
   (b) $29^2$
   (c) $67^2$
   (d) $97^2$
   (e) $21 \times 19$
   (f) $202^2$
   (g) $501^2$
   (h) $999^2$
   (i) $1\,003^2$

(j) 2 998 × 3002

2. Use quadratic identities to find the areas of the rectangles whose dimensions are:

   (a) 33 m by 27 m
   (b) 104 m by 96 m
   (c) 99 m by 101 m
   (d) 998 m by 1 002 m

The following examples illustrate how to expand binomial products using quadratic identities.

### *Example 3.4*

*Perform the indicated multiplication and simplify.*

(a) $(3x + 2)^2$ (b) $(4a - 5b)^2$

### *Solution*

(a) $(3x + 2)^2 = (3x)^2 + 2(3x)(2) + (2)^2$

$= 9x^2 + 12x + 4$

(b) $(4a - 5b)^2 = (4a)^2 + 2(4a)(^-5b) + (^-5b)^2$

$= 16a^2 + (^-40ab) + 25b^2$

$= 16a^2 - 40ab + 25b^2$

### *Example 3.5*

*Perform the indicated multiplication and simplify.*

(a) $(x + 2y)(x - 2y)$

(b) $(3a - b)(3a + b)$

### *Solution*

(a) *In* $(x + 2y)(x - 2y)$, *the two factors are* $(x + 2y)$ *and* $(x - 2y)$.

*Square of first term is* $(x)^2 = x^2$

*Square of second term is* $(2y)^2$ *or* $(^-2y)^2 = 4y^2$

*The difference is* $x^2 - 4y^2$

$(x + 2y)(x - 2y) = x^2 - 4y^2$

(b) $(3a - b)(3a + b) = (3a)^2 - (b)^2$

$= 9a^2 - b^2$

## Exercise 3.4

1. Expand the following using the method of Example 3.4.

   (a) (i) $(a + 1)^2$
   (ii) $(a + 6b)^2$
   (iii) $(x + y)^2$
   (iv) $(x + 9)^2$
   (v) $(m + n)^2$
   (vi) $(2a + 3b)^2$
   (vii) $(3x + 4)^2$
   (viii) $(3m + 2)^2$
   (ix) $(4x + 3y)^2$

   (b) (i) $(b - 1)^2$
   (ii) $(r - 3)^2$
   (iii) $(x - y)^2$
   (iv) $(4x - 3)^2$
   (v) $(5x - 2)^2$
   (vi) $(3x - 12)^2$
   (vii) $(5x - 3)^2$
   (viii) $(4z - 3b)^2$
   (ix) $(7x - 2y)^2$

2. Expand the following using the method of Example 3.5.

   (a) $(a + 3)(a - 3)$
   (b) $(a + 5)(a - 5)$
   (c) $(x - 9)(x + 9)$
   (d) $(f + g)(f - g)$
   (e) $(2p - 1)(2p + 1)$

(f) $(4x - y)(4x + y)$
(g) $(7 + 2x)(7 - 2x)$
(h) $(2a + 3b)(2a - 3b)$
(i) $(5y + 3)(5y - 3)$
(j) $(4x - 1)(4x + 1)$
(k) $(3x + 4)(3x - 4)$
(l) $(2x - 3y)(2x + 3y)$
(m) $(8 - 3x)(8 + 3x)$
(n) $(3x + 7y)(3x - 7y)$

## Quadratic expressions

An algebraic expression of the type $ax^2 + bx + c$ where $a$, $b$ and $c$ are constants, $a \neq 0$ and $x$ is the variable, is called a **quadratic expression** .

Thus, $x^2 + 5x + 6$, $3x^2 - 5x + 3$, $3x^2 + 5x$, $2x^2 - 16$ are examples of quadratic expressions.

In $x^2 + 5x + 6$, the term in $x^2$ is called the **quadratic term** or simply the **first term** . The term in $x$, i.e. $5x$, is called the **linear term** or **second term** or **middle term** , and 6, the numerical term or the term independent of $x$, is called the **constant term** or **third term** .

In $3x^2 + 5x$, the 'missing' constant term is understood to be zero.

In $2x^2 - 16$, the 'missing' linear term has zero coefficient.

## Factorising quadratic expressions

It is easy to see that $2x^2 - 16 = 2(x^2 - 8)$ and $3x^2 - 5x = x(3x - 5)$.

However, it is not easy to see what the factors of $x^2 + 5x + 6$ are. Our experience in multiplying binomials is of great help here.

Now, consider the product $(x + 3)(x + 2)$. $(x + 3)$ and $(x + 2)$ are **prime binomial expressions** , since the two terms in each bracket have no common factor.

$$
\begin{aligned}
(x + 3)(x + 2) &= x(x + 2) + 3(x + 2) \\
&= x^2 + 2x + 3x + 6 \\
&= x^2 + 5x + 6 \text{ (since } 2x \text{ and } 3x \text{ are like terms).}
\end{aligned}
$$

This means that $(x + 3)$ and $(x + 2)$ are factors of $x^2 + 5x + 6$.

$\Rightarrow x^2 + 5x + 6 = (x + 3)(x + 2)$ (in factor form).

**Note:** In $x^2 + 5x + 6$,

1. the coefficient of the quadratic term is 1,
2. the coefficient of the linear term is 5, the sum of the constant terms in the binomial factors, and
3. the constant term is 6, the product of the constant terms in the binomial factors.

In a simple expression like $ax^2 + bx + c$, where $a = 1$, the factors are always of the form $(x + m)(x + n)$, where $m$ and $n$ are constants. Such an expression is factorisable only if there exists two integers $m$ and $n$ such that $m \times n = c$ and $m + n = b$.

To factorise a quadratic expression of the form $ax^2 + bx + c$, where $a = 1$, follow the steps below.

1. List all the possible pairs of integers whose product equals the constant term.
2. Identify the only pair whose sum equals the coefficient of the linear term.
3. Rewrite the given expression with the linear term split as per the factors in 2 above.
4. Factorise your new expression by grouping, i.e. taking two terms at a time.
5. Check that the factors are correct by expanding and simplifying.

## *Example 3.6*

*Factorise* $x^2 + 8x + 12$.

## *Solution*

*In this example,* $a = 1$, $b = 8$ *and* $c = 12$.

1. *List all the pairs of integers whose product is 12. These are:*

   $1 \times 12$

   $3 \times 4$

$2 \times 6$

$^{-}1 \times {}^{-}12$

$^{-}3 \times {}^{-}4$

$^{-}2 \times {}^{-}6$

2. *Identify the pair of numbers whose sum is 8. The numbers are 6 and 2.*
3. *Rewrite the expression with the middle term split.*

   $x + \mathbf{8x} + 12 = x2 + \mathbf{2x} + \mathbf{6x} + 12$
4. *Factorise $x^2 + 2x + 6x + 12$ by grouping. $x^2 + 2x + 6x + 12$ has 4 terms which we can group in twos so that first and second terms make one group and third and fourth terms make another group.*

   i.e $\underbrace{x^2 + 2x} + \underbrace{6x + 12}$

   *In each group, factor out the common factor.*

   *Thus,*

   $x^2 + 2x + 6x + 12 = x(x + 2) + 6(x + 2)$

   *We now have two terms, i.e. $x(x + 2)$ and $6(x + 2)$, whose common factor is $(x + 2)$*

   $\therefore x^2 + 8x + 12 = (x + 2)(x + 6)$ *(Factor out the common factor $(x + 2)$)*

   *Check that $(x + 2)(x + 6) = x^2 + 8x + 12$.*

***Note:*** *Since all the terms in the example are positive, the negative pairs of factors of 12 in 1 above could have been omitted altogether.*

## *Example 3.7*

*Factorise $y^2 + 2y – 35$.*

## *Solution*

*The pairs of numbers whose product is $^{-}35$ are $^{-}5, 7$; $5, {}^{-}7$; $1, {}^{-}35$; and $^{-}1, 35$.*

*The only pair of numbers whose sum is 2 is –5, 7.*

$$\therefore\ y^2 + 2y - 35 = \underbrace{y^2 - 5y} + \underbrace{7y - 35}$$

$$= y(y - 5) + 7(y - 5)$$

$$= (y - 5)(y + 7)$$

**Note:**

1. If the third term in the split form of the expression is negative, we factor out the negative common factor.

   e.g. $y^2 + 2y - 35 = y^2 + 7y - 5y - 35$

   (the third term is negative)

   $= y(y + 7) - 5(y + 7)$ (we factor out $^-5$)

   $= (y + 7)(y - 5)$.

2. The order in which we write $mx$ and $nx$ in the split form of the expression does not change the answer.

## Exercise 3.5

1. Factorise the following by grouping.

   (a) $ax + ay + bx + by$

   (b) $x^2 + 3x + 2x + 6$

   (c) $6x^2 - 9x - 4x + 6$

   (d) $x^2 - 3x - 2x + 6$

   (e) $cx + dx + cy + dy$

   (f) $ax + bx - ay - by$

Factorise the following quadratic expressions:

2. (a) $x^2 + 4x + 3$

   (b) $x^2 + 12x + 32$

   (c) $x^2 + 20x + 100$

   (d) $x^2 + 11x + 18$

   (e) $x^2 + 3x + 2$

   (f) $x^2 + 6x + 9$

3. (a) $x^2 + 5x - 24$

   (b) $x^2 + 2x - 63$

   (c) $x^2 + x - 12$

   (d) $x^2 + 2x - 15$

   (e) $x^2 + x - 6$

   (f) $x^2 + 5x - 6$

4. (a) $x^2 - 8x + 15$
   (b) $x^2 - 9x + 14$
   (c) $x^2 - 2x + 1$
   (d) $x^2 - 4x + 4$
   (e) $x^2 - 10x + 24$
   (f) $x^2 - 6x + 9$

5. (a) $x^2 - x - 12$
   (b) $x^2 - 5x - 24$
   (c) $x^2 - x - 30$
   (d) $x^2 - 3x - 18$
   (e) $x^2 - 3x - 10$
   (f) $x^2 - x - 20$

## Further factorisation of quadratic expressions

Consider the product $(2x + 3)(2x + 7)$.

$$
\begin{aligned}
(2x + 3)(2x + 7) &= 2x(2x + 7) + 3(2x + 7) \\
&= 4x^2 + 14x + 6x + 21 \\
&= 4x^2 + (14 + 6)x + 21 \\
&= 4x^2 + 20x + 21
\end{aligned}
$$

In this example, $(2x + 3)$ and $(2x + 7)$ are the factors of $4x^2 + 20x + 21$. In $4x^2 + 20x + 21$, $a = 4$, $b = 20$, and $c = 21$.

**Note:**

1. $ac = 4 \times 21 = 84$
2. $b = 20$
3. There is a pair of integers $m$ and $n$ such that $m \times n = ac$ and $m + n = b$. The pair is 14 and 6.

An expression of the form $ax^2 + bx + c$ can be factorised if there exists a pair of numbers $m$ and $n$ whose product is $ac$ and whose sum is $b$.

### *Example 3.8*

*Factorise the quadratic expression* $6x^2 + 13x + 6$.

### *Solution*

*In this example,* $a = 6$, $b = 13$ *and* $c = 6$.

*Step 1: Determine if the expression is factorisable.*

$ac = 36$.

*Possible pairs of m and n are*

$2 \times 18$, $3 \times 12$, $4 \times 9$, $6 \times 6$, $1 \times 36$

*Since all the terms are positive, we will only consider positive values of m and n.*

*Step 2:* $6x^2 + 13x + 6 = 6x^2 + 9x + 4x + 6$

*(Split middle term)*

*Step 3:* $= \ = \overline{6x^2 + 9x} + \overline{4x + 6}$ *(Group in pairs)*

$= 3x(2x + 3) + 2(2x + 3)$ *(Factorise the groups)*

*common factor*

*Step 4:* $= (2x + 3)(3x + 2)$ *(Factor out the common factor)*

*Step 5: Check if the factors are correct by expanding* $(2x + 3)(3x + 2)$ *and simplifying.*

**Note:** When you factorise the groups in Step 3, the factors inside the brackets must be identical. If not, then there is a mistake.

### *Example 3.9*

*Factorise* $12x^2 - 4x - 5$.

### *Solution*

*In this example,* $a = 12$, $b = {}^{-}4$, $c = {}^{-}5$ *and* $ac = {}^{-}60$.

*By inspection, m and n are* ${}^{-}10$ *and* $6$.

$\therefore 12x^2 - 4x - 5 = 12x^2 - 10x + 6x - 5$

$2x(6x - 5) + 1(6x - 5)$

*common factor*

$= (6x - 5)(2x + 1)$

**Note:**

1. If we cannot determine $m$ and $n$ by inspection, then we use the procedure of Example 3.8.
2. If $m$ and $n$ do not exist, then the expression has no factors.

## Exercise 3.6

Factorise the following expressions:

1. $x(x + 1) + 3(x + 1)$
2. $3(2x + 1) - x(2x + 1)$
3. $4a(2a - 3) - 3(2a - 3)$
4. $4b(b + 6) - (b + 6)$
5. $3y(4 - y) + 6(4 - y)$
6. $2x^2 + x - 6$
7. $3a^2 + 7a - 6$
8. $2x^2 + 3x + 1$
9. $4x^2 - 2x - 6$
10. $4y^2 - 4y - 3$
11. $9b^2 - 21b - 8$
12. $7x^2 - 3x + 6$
13. $2x^2 + 6x - 20$
14. $6x^2 + 5x - 6$
15. $15a^2 + 2a - 1$
16. $9a^2 + 21a - 8$

17. $8b^2 - 18b + 9$
18. $10a^2 + 9a + 2$
19. $7x^2 - 36x + 5$
20. $6x^2 + 23x + 15$

## Perfect squares

Just like we have square numbers in arithmetic, we also have square trinomials in algebra.

Remember $(a + b)^2 = (a + b)(a + b)$

$= a^2 + 2ab + b^2$

In this case, $a^2 + 2ab + b^2$ is a perfect square. It has two identical factors.

If a trinomial is a perfect square,

1. The first term must be a perfect square.
2. The last term must be a perfect square.
3. The middle term must be twice the product of numbers that were squared to give the first and last terms.

***Example 3.10***

*Show that the following expressions are perfect squares and give the factor of each.*

(a) $9x^2 + 12x + 4$

(b) $9x^2 - 30x + 25$

***Solution***

(a) $9x^2 + 12x + 4$

Condition (1): first term $9x^2 = (3x)^2$

Condition (2): last term $4 = (2)^2$

*Condition (3): middle term* $12x = 2(3x)(2)$

$\therefore 9x^2 + 12x + 4 = (3x)^2 + 2(2)(3x) + 2^2$

$= (3x + 2)^2$

(b) $9x^2 - 30x + 25$

*First term* $9x^2 = (3x)^2$

*Last term* $25 = (^-5)^2$

*Middle term* $^-30x = 2(^-5)(3x)$

$\therefore 9x^2 - 30x + 25$ *is a perfect square which factorises to* $(3x - 5)^2$.

***Note:*** *In* $9x^2 - 30x + 25$*, middle term of the expression is negative, hence the constant term in the binomial factor must be negative.*

## Exercise 3.7

Show that the following are perfect squares. Hence state their factors.

1. $x^2 + 8x + 16$
2. $x^2 + 12x + 36$
3. $x^2 - 14x + 49$
4. $y^2 - 6y + 9$
5. $4x^2 + 20x + 25$
6. $9x^2 - 42x + 49$
7. $9x^2 - 6x + 1$
8. $16x^2 + 24x + 9$
9. $25x^2 - 40xy + 16y^2$
10. $144x^2 - 120x + 25$
11. $4x^2 + 12x + 9$
12. $36x^2 - 108x + 81$

## Factorising a difference of two squares

**Remember:** We have already seen that $(a - b)(a + b) = a^2 - b^2$ .

$(a - b)(a + b)$ is the product of the sum and difference of the same two terms. The product always gives a difference of the squares of the two terms.

To factorise a difference of two squares, we reverse the process, i.e. find the factors, given the expression.

In order to use this technique, we must be able to recognise a difference of two perfect squares. We proceed as in Example 3.11.

To factorise a difference of two squares, follow the following steps:

Step 1: Confirm that we have a perfect square minus another perfect square.

Step 2: Rewrite the expression in the form $a^2 - b^2$ .

Step 3: Factorise the expression.

## *Example 3.11*

*Factorise*

(a) $x^2 - 9$

(b) $4x^2 - 25y^2$

(c) $3x^2 - 27$

## *Solution*

(a) In $x^2 - 9$, $x^2$ and 9 are perfect squares.

$$\therefore x^2 - 9 = (x)^2 - (3)^2$$
$$= (x + 3)(x - 3)$$

(b) In $4x^2 - 25y^2$ , $4x^2$ and $25y^2$ are perfect squares.

$$\therefore 4x^2 - 25y^2 = (2x)^2 - (5y)^2$$
$$= (2x + 5y)(2x - 5y)$$

(c) In $3x^2 - 27$, $3x^2$ and 27 are not perfect squares but they have a common factor.

$$\therefore 3x^2 - 27 = 3(x^2 - 9) \text{ (} x^2 \text{ and 9 are perfect squares)}$$
$$= 3[(x)^2 - (3)^2]$$
$$= 3[(x - 3)(x + 3)]$$
$$= 3(x - 3)(x + 3)$$

**Note** that in $3x^2 - 27$, it was necessary for us to factor out the common factor 3 in order to discover the difference of two squares therein. We must not forget to include 3 in our answer.

Also note that an expression of the form $\boldsymbol{a^2 + b^2}$ is called the **sum of two squares** , and **it has no factors** .

## Exercise 3.8

Factorise the following completely.

1. (a) $x^2 - 16$
   (b) $x^2 - 4$
   (c) $x^2 - 25$
2. (a) $x^2 - 1$
   (b) $36 - a^2$
   (c) $81 - a^2$
3. (a) $25 - y^2$
   (b) $x^2 - y^2$
   (c) $x^2 - 4y^2$
4. (a) $b^2 - 49$
   (b) $4a^2 - 25b^2$
   (c) $9x^2 - 49y^2$
5. (a) $9y^2 - 25x^2$
   (b) $16p^2 - 9q^2$
   (c) $4x^2 - 9b^2$
6. (a) $81x^2 - y^2$
   (b) $p^2 - 25q^2$
   (c) $a^2 - 16b^2$

7. (a) $144x^2 - 121y^2$
   (b) $1 - c^2$
   (c) $2x^2 - 8y^2$
8. (a) $3x^2 - 48y^2$
   (b) $18x^2 - 2$
   (c) $20 - 5b^2$
9. (a) $8x^2 - 32y^2$
   (b) $50 - 2x^2$
   (c) $r^4 - 9$
10. (a) $49x^2 - 64y^4$
   (b) $x^4 - 1$
   (c) $a^4 b^4 - 16c^4$

## Simplifying algebraic fractions

In Form 1, we learnt that,

1. Dividing or multiplying the numerator and denominator of an algebraic fraction by the same number does not change the value of the fraction.
2. When combining fractions, we first express them over a common denominator by finding the LCM of the denominators.

### *Example 3.12*

*Simplify the following fractions to the lowest terms.*

*(a)* $\frac{bx}{b^2}$

*(b)* $\frac{6a^2d}{15ad^2}$

### *Solution*

*(a)* $\frac{bx}{b^2}$ *(Divide the numerator and the denominator by b)*

*We get,*

$$\frac{bx}{b^2} = \frac{\cancel{b} \times x}{\cancel{b} \times b}$$

$$= \frac{x}{b}$$

(b) $\frac{6a^2d}{15ad^2}$ *(Divide the numerator and the 15ad denominator by 3ad)*

*We get,*

$$\frac{6a^2d}{15ad^2} = \frac{\overset{2}{\cancel{6}} \times a \times \cancel{a} \times \cancel{d}}{\underset{5}{\cancel{15}} \times \cancel{a} \times \cancel{d} \times d}$$

$$= \frac{2a}{5d}$$

## *Example 3.13*

*Simplify the following algebraic fractions.*

(a) $\frac{2y}{x} + \frac{3}{2x} + \frac{4y}{3x}$

(b) $\frac{3x - 5y}{6z} - \frac{2x - 7y}{2z}$

(c) $\frac{2x^2z}{9xz^2} \div \frac{3x^3}{8x^3z^2}$

(d) $2\frac{1}{7}q$ of $\frac{14r^2}{3q^3}$

## *Solution*

(a) $\frac{2y}{x} + \frac{3}{2x} + \frac{4y}{3x}$

*LCM of x, 2x and 3x is 6x*

$$= \frac{12y + 9 + 8y}{6x} \quad \textit{(Add like terms)}$$

$$= \frac{20y + 9}{6x}$$

(b) $\frac{3x - 5y}{6z} - \frac{2x - 7y}{2z}$

LCM of 6z and 2z is 6z

$$= \frac{(3x - 5y) - 3(2x - 7y)}{6z} \quad \textit{(Remove brackets)}$$

$$= \frac{3x - 5y - 6x + 21y}{6z} \quad \textit{(Subtract like terms)}$$

$$= \frac{^{-}3x + 16y}{6z} = \frac{16y - 3x}{6z}$$

(c) $\frac{2x^2z}{9xz^2} \div \frac{3x^3}{8x^3z^2}$

$$= \frac{2x^{\cancel{2}}z}{9\cancel{x}\cancel{z^2}} \times \frac{8\cancel{x^3}\cancel{z^2}}{3x^{\cancel{3}}} \quad \textit{(Multiplying by the multiplication inverse of the divisor)}$$

$$= \frac{16xz}{27}$$

(d) $2\frac{1}{7}q \text{ of } \frac{14r^2}{3q^2}$

$$= \frac{\overset{5}{\cancel{15q}}}{\cancel{7}} \times \frac{\overset{2}{\cancel{14}}r^2}{\cancel{3q^2}} = \frac{10r^2}{q}$$

## Exercise 3.9

1. Simplify the following fractions.
   (a) $\frac{99y^2}{132y}$
   (b) $\frac{108xy^2}{144x^2y}$

(c) $\frac{42p^2}{63p}$

2. Simplify

(a) $\frac{p+3}{3} + \frac{2x}{3}$

(b) $\frac{1}{x} + \frac{1}{x+1}$

(c) $\frac{y+x}{x} + \frac{y}{2x}$

(d) $\frac{x+3}{6} + \frac{x+5}{4}$

(e) $\frac{a+b}{c} - \frac{a-c}{b}$

(f) $\frac{2x-4y}{5z} - \frac{2x+6y}{z}$

(g) $\frac{x-1}{x} - \frac{2x+1}{3x}$

(h) $\frac{mn}{4} - \frac{1}{3}my$

3. Simplify

(a) $3\frac{1}{2}p$ of $\frac{15q}{8p^2}$

(b) $3\frac{3}{5}xy^2z \times \frac{5}{6yt}$

(c) $\frac{9pqr}{10p} \times \frac{16m}{21q^2}$

(d) $\frac{m}{5}$ of $\frac{15n}{16m^2}$

(e) $\frac{16fg^2h}{27} \div \frac{8g}{45}$

(f) $\frac{8abc}{15p^2q} \div \frac{24c}{25p}$

(g) $\frac{2x-2}{x^2} \div \frac{x-1}{x}$

(h) $\frac{2x^2}{3} \div \frac{4}{9xy}$

## Solving linear equations involving fractions

When solving equations involving fractions, simplify the fractions using the LCM.

### *Example 3.14*

*Solve the following equations.*

(a) $\frac{2x+7}{3} - \frac{5x+6}{4} = 0$

(b) $\frac{2a+36}{a} - \frac{4}{5} = 0$

### *Solution*

(a)

$$\frac{2x+7}{3} - \frac{5x+6}{4} = 0$$

$$\frac{4(2x+7) - 3(5x+6)}{12} = 0 \quad \text{(Use LCM to get same denominator)}$$

$$4(2x+7) - 3(5x+6) = 0 \quad \text{(Multiply both sides by 12)}$$

$$8x + 28 - 15x - 18 = 0 \quad \text{(Open brackets)}$$

$$^{-}7x + 10 = 0$$

$$^{-}7x = {}^{-}10$$

$$\therefore\ x = \frac{10}{7} = 1\frac{3}{7}$$

(b)

$$\frac{2a+36}{a} - \frac{4}{5} = 0$$

$$\frac{5(2a+36) - 4a}{5a} = 0$$

$$5(2a+36) - 4a = 0$$

$$10a + 180 - 4a = 0$$

$$6a + 180 = 0$$

$$6a = {}^{-}180$$

$$a = {}^{-}30$$

## Exercise 3.10

Solve the following equations.

1. (a) $\frac{x}{3} + 3 = 3$

   (b) $^{-}3 - \frac{x}{2} = 4$

   (c) $\frac{k}{5} = 0$

   (d) $p - 2\frac{1}{2} = 6\frac{1}{2}$

2. (a) $^{-}3\frac{3}{4} = x + 1\frac{2}{5}$

   (b) $4\frac{1}{2} = 5q - \frac{1}{4}$

3. (a) $3\frac{1}{2} + 2\frac{1}{4}f = 17\frac{1}{2} - 1\frac{1}{4}f$

   (b) $2\frac{1}{2}x - \frac{1}{4} = 1\frac{1}{2}x + 3\frac{1}{4}$

   (c) $\frac{0.1}{x} + \frac{3.9}{x} = 12$

   (d) $\frac{12}{x} - \frac{1}{2} = 1\frac{1}{3} + \frac{3}{x}$

4. $\frac{1}{2+x} = \frac{3}{2x+5}$

5. (a) $\frac{1}{x} = 2\frac{1}{2}$

   (b) $\frac{1}{y-1} = \frac{2}{7}$

   (c) $\frac{2}{p-1} = \frac{5}{p}$

# 4 PLANE SHAPES

## Quadrilaterals

A **quadrilateral** is a polygon with four sides and four angles. Fig. 4.1 shows some quadrilaterals.

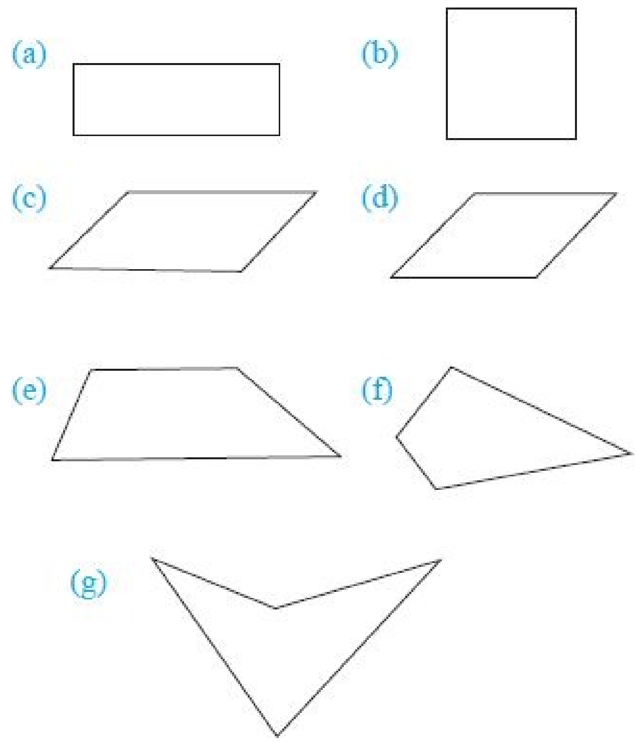



*Fig. 4.1*

## Rectangles and squares

## Rectangle

A **rectangle** is a quadrilateral in which opposite sides are equal and parallel, and all angles are right angles (Fig. 4.2 ).

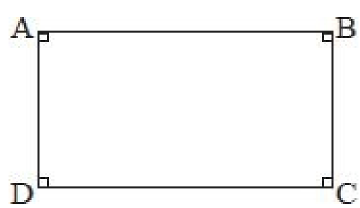


*Fig. 4.2*

## Square

A **square** is a rectangle in which all sides have the same length (Fig. 4.3 ).

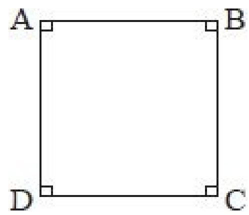



*Fig. 4.3*

## Exercise 4.1

1. Name 5 rectangular shapes in your classroom.
2. Copy rectangle ABCD (Fig. 4.4 ). Join AC and BD.


*Fig. 4.4*

(a) Measure AC and BD. What do you notice?
(b) AC and BD meet at point O. Measure AO, OC, BO and OD. What do you notice?

**Note:** AC and BD are called **diagonals** of rectangle ABCD and O is the centre of the rectangle.

3. Consider rectangle PQRS (Fig. 4.5 ). O is a point inside the rectangle.

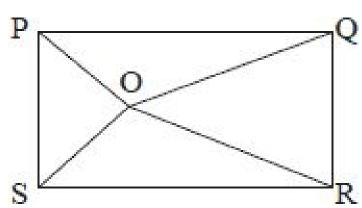



*Fig. 4.5*

(a) How many triangles are there?

(b) What is the sum of the angles of all the triangles in (a) ?
(c) What is the sum of the angles at O?
(d) Subtract the answer you got in (c) from the one you got in (b) . What can you say about the sum of the angles of a rectangle?

4. Consider a rectangle with sides extended as in Fig. 4.6 . What is the sum of the marked exterior angles?

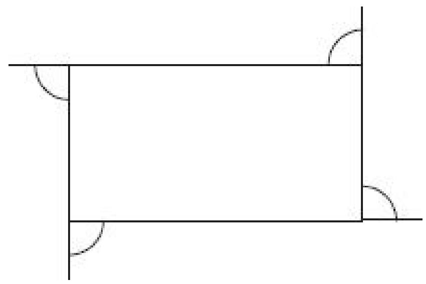


*Fig. 4.6*

5. Copy the square in Fig 4.7 . Draw the diagonals of the square.

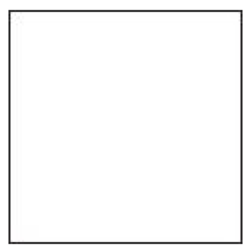


*Fig. 4.7*

(a) Measure the diagonals. What can you say about their lengths?
(b) What does each diagonal do to the angles of the square?
(c) What is the size of the angles formed where the diagonals meet?

From Exercise 4.1, you should have noticed the following.

For a rectangle,

(i) diagonals of a rectangle are equal in length.
(ii) diagonals of a rectangle bisect each other (i.e. they cut each other into two equal parts).
(iii) the sum of the interior angles of a rectangle is 360º or 4 right angles.

(iv) the sum of the exterior angles of a rectangle is 360º or 4 right angles.

For a square,

(i) the diagonals of a square bisect the angles of the square.
(ii) diagonals of a square intersect at right angles.

## Parallelograms, rhombi and trapezia

### Parallelogram

A **parallelogram** is a quadrilateral in which opposite sides are equal and parallel (Fig. 4.8 ).

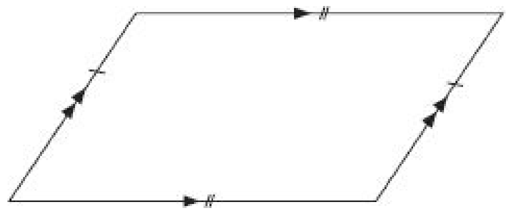


*Fig. 4.8*

### Rhombus

A **rhombus** (plural: rhombi) is a parallelogram with all the four sides equal in length (Fig. 4.9 ).

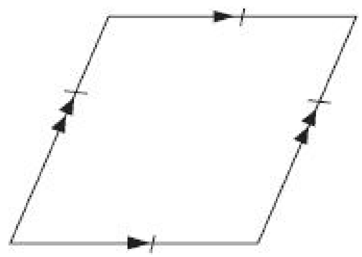


*Fig. 4.9*

### Trapezium

A **trapezium** (plural: trapezia) is a quadrilateral which has at least one pair of opposite sides parallel (Fig. 4.10 ). If the remaining two sides are equal, the trapezium is said to be isosceles.

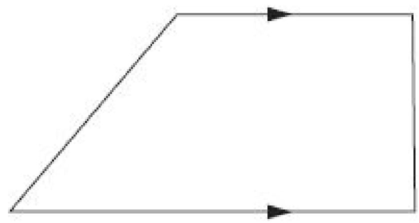


*Fig. 4.10*

## Exercise 4.2

1. Fig. 4.11 is a quadrilateral. Measure all its angles and sides.

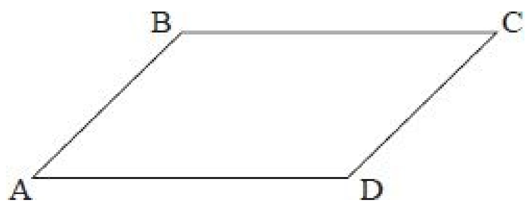



*Fig. 4.11*

   (a) What can you say about the angles?
   (b) What can you say about the sides?
   (c) What is the name of the quadrilateral ABCD?

2. Copy the quadrilateral in Fig. 4.11 . Draw the diagonals of the quadrilateral to meet at point O.
   (a) Name all the pairs of equal angles. State the reasons why the angles in each pair are equal.
   (b) Measure AO, BO, CO and DO. What do you notice?

3. Fig 4.12 shows a parallelogram with sides produced.

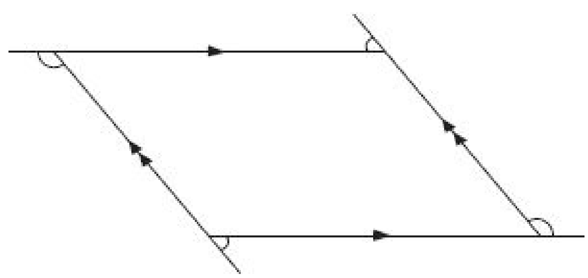


*Fig. 4.12*

   What is the sum of the marked exterior angles?

4. Make a rhombus from a sheet of paper using the following method.

(a) Fold a rectangular sheet of paper such that the opposite sides correspond (Fig. 4.13(i)).
(b) Fold the paper again as in Fig. 4.13(ii).
(c) Draw a line as shown in Fig. 4.13(ii).
(d) Cut the paper along the dotted line. Unfold the triangular part. This gives a rhombus (Fig. 4.13(iii)).

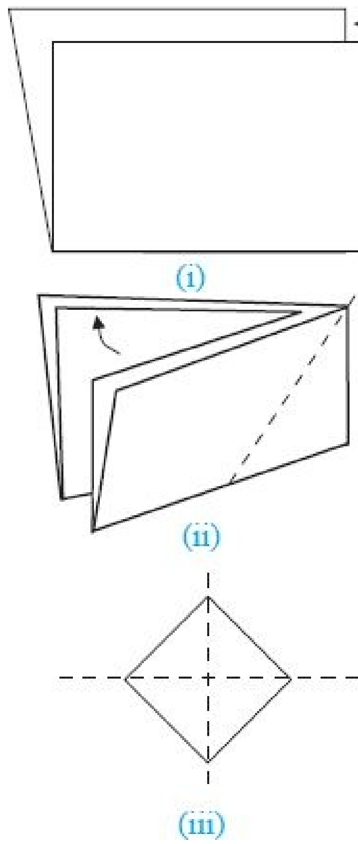


*Fig. 4.13*

5. Using the rhombus you made in Question 4,
   (a) measure all the sides. What do you notice?
   (b) measure all the angles. What do you notice?

6. Draw a rhombus (by tracing the one you made in Question 4). Draw the diagonals.
   (a) What angles are formed where the diagonals meet?
   (b) Name all angles that are equal.
   (c) What is the sum of all the angles of the rhombus?

7. (a) State 2 differences between

(i) a square and a rectangle
(ii) a square and a parallelogram
(iii) a rectangle and a parallelogram

(b) Is a rectangle a parallelogram?

8. (a) What properties are common between a square and a rhombus?

    (b) State any differences between a rhombus and a square.
    (c) Is a square a rhombus?

9. (a) What differences are there between a trapezium and a parallelogram?

    (b) Is a parallelogram a trapezium?
    (c) List 3 objects which have the shape of a trapezium.

From Exercise 4.2, you should have noticed the following.

In a parallelogram,

(i) opposite sides are equal in length.
(ii) opposite angles are equal.
(iii) the sum of interior angles is 360º.
(iv) the sum of exterior angles is 360º.
(v) the diagonals bisect each other.

In a rhombus (which is a special parallelogram, with all sides equal), the diagonals

(i) bisect each other at right angles.
(ii) bisect the angles of the rhombus.

## Kite

A **kite** is a quadrilateral with pairs of adjacent equal sides.

## Exercise 4.3

1. Draw triangles ABC and ABD accurately as shown in Fig. 4.14 .

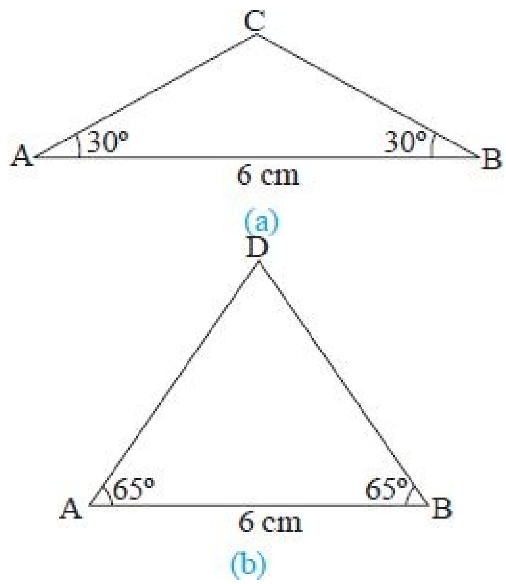



*Fig. 4.14*

2. Cut out the two triangles and place them base to base, i.e. with sides AB together as shown in Fig. 4.15 .

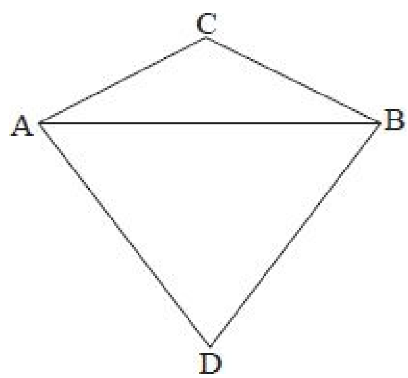



*Fig. 4.15*

Copy the resulting figure ACBD in your exercise book.

(a) Measure the sides of figure ACBD. What can you say about the sides of the figure?

(b) Join the diagonals AB and CD and let them meet at a point O. What can you say about them?

A quadrilateral similar to ACBD (Fig. 4.15 is called a **kite** .

Is a rhombus a kite?

3. In Fig. 4.15 measure

(a) <ACD and <BCD. What do you notice?

(b) <ADC and <BDC. What do you notice?
(c) Measure all the angles formed at point O. What do you notice?
(d) Measure line segments OA, OB, OC and OD. What do you notice?

From Exercise 4.3, you should have noticed that

(i) two pairs of adjacent sides of a kite are equal in length.
(ii) the longer diagonal of a kite bisects the shorter one at right angles.
(iii) the longer diagonal of a kite bisects the angle at the vertices through which it is drawn.

## Theorems on parallelograms

In Exercise 4.2, we deducted that for a parallelogram,

1. the opposite sides are equal.
2. the opposite angles are equal.
3. the diagonals bisect each other.

This deductions are referred to as **parallelogram theorems** . A theorem is a statement that can be proven to be true by way of reasoning. Thus, we will now prove each of the statements to be true.

## Proof of parallelogram theorems

### Opposite sides are equal

Given a parallelogram PQRS, prove that PQ = SR and PS = QR.

*Proof:* Draw parallelogram PQRS (Fig. 4.16 ).

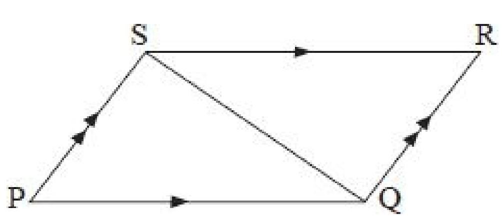



*Fig. 4.16*

Join SQ.

In Δs PQS and QSR,

∠PQS = ∠QSR (alternate angles, PQ//SR)

∠PSQ = ∠SQR (alternate angles, PS//QR)

SQ = QR (common line)

∴ ΔPQS and ΔQSR are congruent (AAS or ASA).

Hence, PQ = SR and PS = QR.

## Opposite angles are equal

Using the parallelogram in Fig 4.16 , prove that ∠SPQ = ∠SRQ and ∠PSR = ∠PQR.

*Proof:* ∠SPQ + ∠PSR =180º (supplementary angles, PQ // SR)

∠SPQ + ∠PQR =180º (supplementary angles, PS // QR)

Thus, ∠SPQ + ∠PSR = ∠SPQ + ∠PQR = 180º

hence, ∠PSR = ∠PQR

Similarly, ∠SPQ = ∠SRQ

## Diagonals bisect each other

Given a parallelogram PQRS with diagonals SQ and PR intersecting at point T (Fig. 4.17 ), prove that ST = TQ and PT = TR.

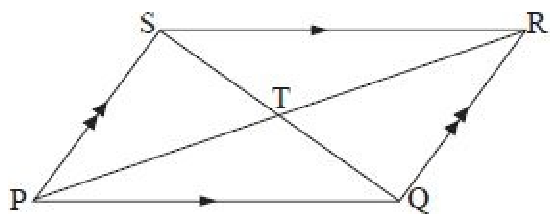



*Fig. 4.17*

*Proof:* In Δs STR and PTQ,

∠RST = ∠TQP (alternate angles, SR // PQ)

∠TRS = ∠TPQ (alternate angles, SR // PQ)

SR = PQ (opposite sides of a parallelogram)

∴ STR and ΔPTQ are congruent (ASA)

hence, ST = TQ and PT = TR.

## Exercise 4.4

1. The interior angles of a quadrilateral are x, 2x, 3x, 4x in that order. Find the angles in degrees.
2. ABCD is a parallelogram. <ADC = $38^\circ$. Find the sizes of the other angles of the parallelogram.
3. PQRS is a kite (Fig. 4.18 ).

   <PSR = $32^\circ$ and <QPS = $102^\circ$.

   Find

   (a) <PQR
   (b) <QRS

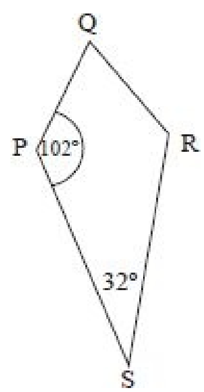



*Fig. 4.18*

4. Fig. 4.19 shows a trapezium KLMN. <KLM = $48^\circ$, <LKM = $110^\circ$ and <KMN = $46^\circ$. Find

   (a) <LMK
   (b) <KNM

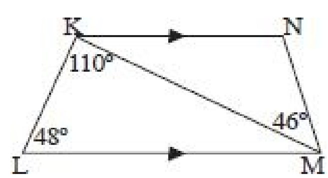



*Fig. 4.19*

5. In quadrilateral ABCD (Fig. 4.20 ) AB = DC, AD = BC, <ADC = $88^\circ$ and <DAC = $74^\circ$. Find x.

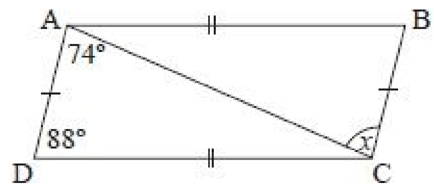



*Fig. 4.20*

6. PQRS is a trapezium with <QPS = 58º (Fig. 4.21 ). A bisector of <PSR cuts PQ at point T. Find <PTS.

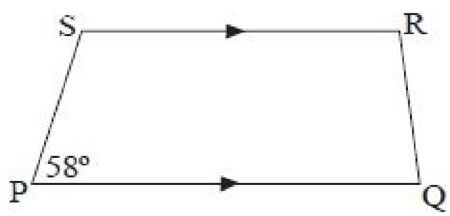



*Fig. 4.21*

# 5 MENSURATION

## Definition of a solid

A solid is a figure that has length, breadth (or width) and thickness. In other words, a solid is a figure that is not a plane (or flat) and therefore occupies space, e.g. a cube or pyramid.

The three measures of a solid namely length, breadth and thickness are called its dimensions.

Many solids are found in nature. If common salt is examined through a magnifying glass, it will be seen that its crystals are cubes. The crystals of graphite are prisms and those of diamond are octahedra.

We also find many man-made geometric solids everywhere. The following are some examples (Fig. 5.1 ).

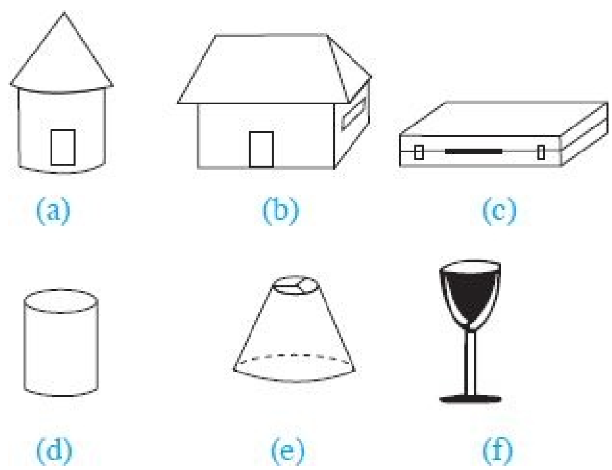



*Fig. 5.1*

(a) and (b) are houses

(c) is a brief-case

(d) is a can of cooking fat

(e) is a lamp shade

(f) is a wine glass

## Naming solids

Solids with many plane faces are called polyhedra. The singular is polyhedron, which means ‘having many faces.’

The following are names and descriptions of some common solids.

1. **Cuboid**

   It is a solid bounded by three pairs of identical faces which are all rectangles.

   Sometimes, a cuboid is referred to as a ‘rectangular box’ or a ‘rectangular block.’

2. **Cube**

   It is a solid bounded by six identical faces which are all squares.

   A cube is a special type of cuboid, and it may sometimes be called a ‘square box’.

3. **Pyramid**

   It is a solid figure with triangular slanting faces which meet at one point, called an apex or vertex, above a polygonal base.

   A pyramid is always named after the shape of its base. Thus, there are triangular-based pyramids (tetrahedra), rectangular-based pyramids, pentagonal-based pyramids, and so on.

   If the pyramid has its vertex vertically above the centre of the base, it is called a right pyramid.

4. **Tetrahedron**

   This is a solid figure with four faces, all triangles. Therefore, it is a pyramid with a triangular base.

5. **Prism**

   It is a solid figure with identical and parallel end faces. Its other faces are parallelograms. If we cut a prism through a plane (flat surface) parallel to an end, the cut surface (cross-section) will be exactly the same as the end face.

   A prism is named according to the shape of the end face. If the end is a triangle, it is a triangular prism; if rectangular, it is a rectangular prism, and so on.

6. **Cone**

   It is a solid figure which narrows to a point (a vertex) from a circular flat base. It is a pyramid with a circular base.

7. **Frustum or frustrum**

   When a cone or pyramid is cut along a plane parallel to the base, the bottom part is referred to as a frustum (or frustrum).

8. **Cylinder**

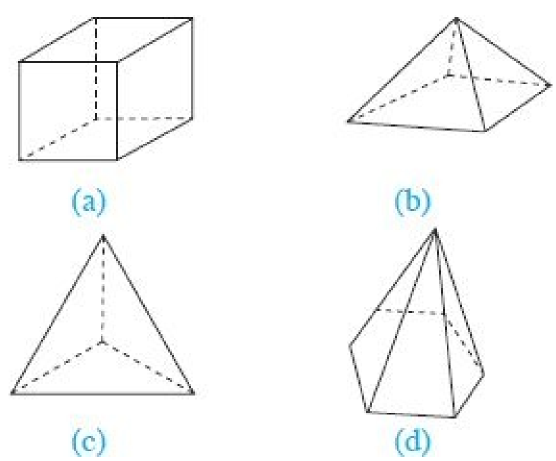
   This is a hollow or solid figure with straight sides and circular ends. It comprises of two identical circular end faces and a curved surface. A cylinder is a prism with a circular cross-section.

9. **Sphere**

   This is a solid figure that is entirely round, such as a ball. Half of a sphere is called a hemisphere.

## Identify them now

Fig. 5.2 shows all the solids described above. Which is which?



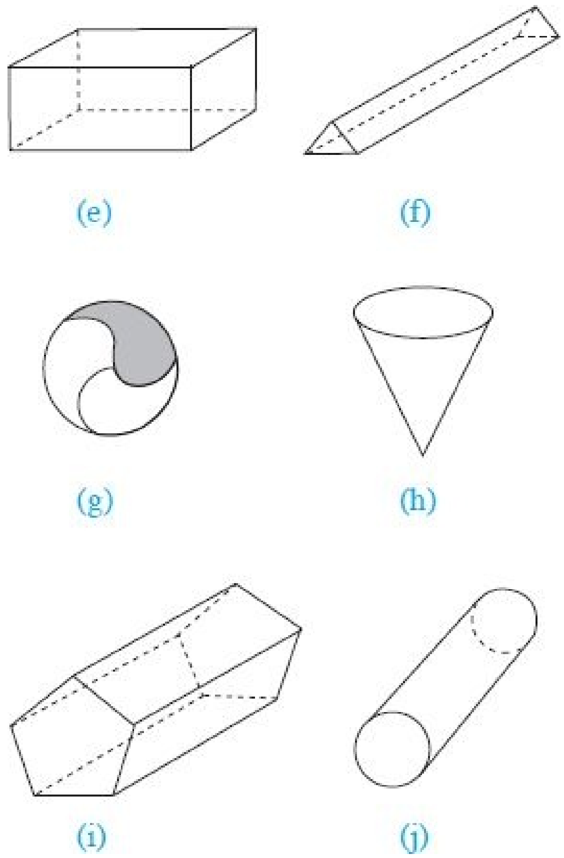



*Fig. 5.2*

Which of these solids are polyhedra?

List all the prisms in Fig. 5.2 .

List all the pyramids in the Fig. 5.2 .

## Nets of solids

If a cardboard box is cut as shown in Fig. 5.3 (a), then the sides and top can be laid out flat, as in Fig. 5.3 (b). The flat shape thus obtained is called a net of the cardboard box. Given a net, we can fold it up to make the outside surface of the relevant solid.

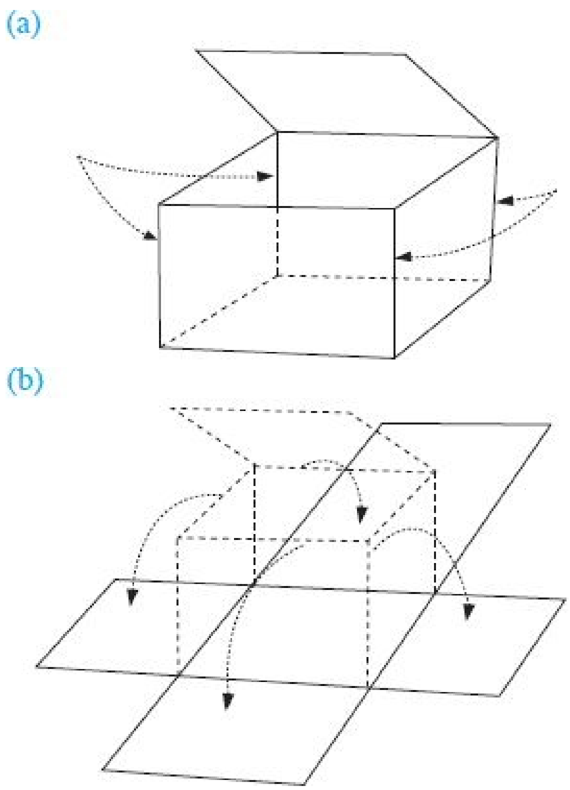



*Fig. 5.3*

How does the net of a cylinder look like?

Fig. 5.4 shows how a cylinder can be cut and laid flat to give its net. The net is made up of two circles and one rectangle.

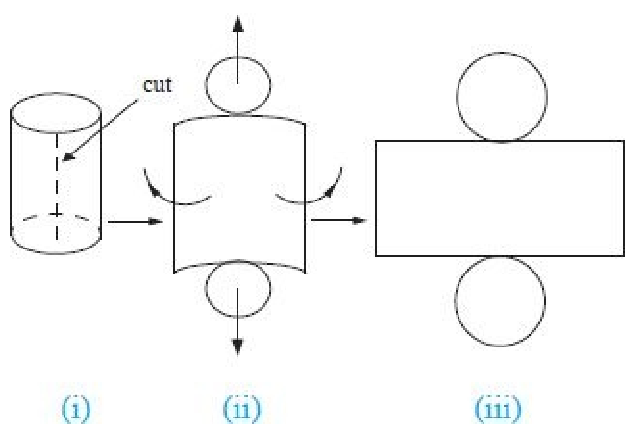



*Fig. 5.4*

## Models of solids

A model of a solid can be constructed by folding up the net of that solid. Nets for making models must be drawn accurately. Graph or squared paper is useful for making an accurate net.

## Procedure of making models of solids

Draw the required net on squared paper. Prick through the squared paper onto the material that you are using for the model. This enables you to have an outline of the net on the material.

Cardboards (like for a carton box) or stiff manila papers are suitable materials for making models.

Before folding, use a pair of compasses point or a knife to score the lines to be folded. In this way, you will obtain clean folds.

You may secure the edges of the solid with adhesive tape. However, you will obtain a better result if you leave tabs on the edges and stick the model together with a quick drying glue. If you cannot decide where to put the tabs, leave them on all free edges and cut them off when not needed.

**Important:** Keep one face free of tabs and stick that one last.

Fig. 5.5 shows the net of a right square-based pyramid.

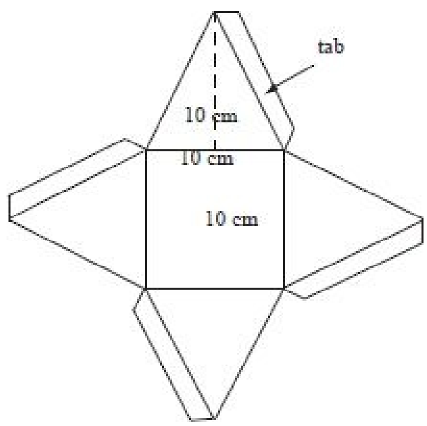



*Fig. 5.5*

### *Activity 5.1*

On a sheet of paper,

(i) Draw two circles of radius 3.5 cm each.
(ii) Draw a rectangle measuring 22 cm by 10 cm.
(iii) Cut-out the circles and rectangle and use them to form a cylinder.

## Surface area of solids from nets

The surface area of a solid is the sum of the areas of all its faces.

## Surface area of a cylinder

Consider the closed cylinder in Fig. 5.6 (a).
What kind of figure is each circular face?
Find the area of the circular face.
If the curved surface is unfolded, what figure is formed (Fig. 5.6 (b))?

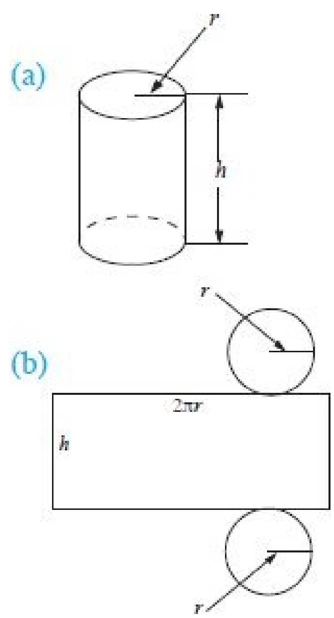



*Fig. 5.6*

Do you see that the length of the rectangle is the circumference of the circular end face?
Find the area of the rectangle.
What is the surface area of the cylinder?
You should have found that:
Area of a circular face = Area of a circle

$= \pi r^2$

Area of the rectangle = Circumference × height

$= 2\pi r \times h$

∴ Surface area of a closed cylinder is given by the area of two circular faces plus the product of the circumference and the height, i.e.

Surface area = $2\pi r^2 + 2\pi rh$

What is the surface area of an open cylinder? (Hint: closed on one end)

What is the surface area of a hollow cylinder? (Hint: open on all ends)

Surface area of a closed cylinder

$= 2\pi r^2 + 2\pi rh$

Surface area of an open cylinder

$= \pi r^2 + 2\pi rh$

Surface area of a hollow cylinder

$= 2\pi rh$

We have seen that to find the surface area of a solid from the net, we calculate the areas of the individual shapes that comprise the net and find their sum.

## Exercise 5.1

1. Use the value $\frac{22}{7}$ for $\pi$ to calculate the surface area of the shapes formed when the nets in Fig. 5.7 are folded up.

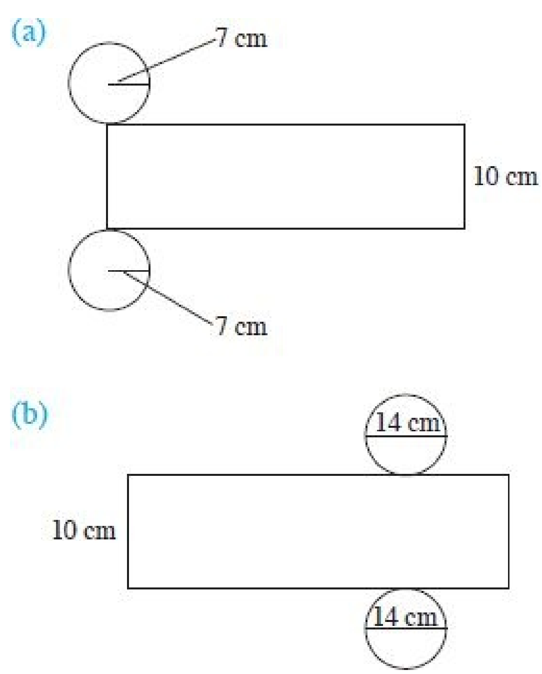



*Fig. 5.7*

2. Fig. 5.8 shows the net of an open cylinder.

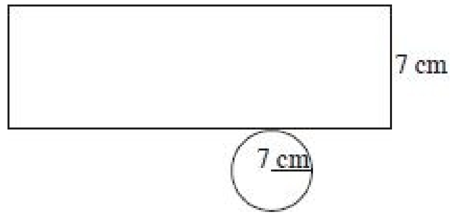



*Fig. 5.8*

(a) If the net is folded to make the cylinder,

(i) what is the radius of the cylinder?

(ii) what is the height of the cylinder?

(b) Use $\pi = \underline{22}$ to find the surface area of the open cylinder.

3. The curved surface area of a cylinder is 1 584 $cm^2$ . If the cylinder has a radius 9 cm, find its height. (Use $\pi = \underline{22}$ ).

4. The radius of a hollow cylinder is 3.5 cm and the height of the cylinder is 6 cm. A cut is made straight down the surface so that the cylinder can be unwrapped to form a rectangle. Using $\pi = \underline{22}$ find the,

   (a) width
   (b) length
   (c) area of the rectangle

5. Use $\pi = 3.14$ to find the surface area of closed tomato paste tins of different sizes given

   (a) circumference = 246.4 cm, height = 8 cm
   (b) diameter = 14.7 cm, height = 0.16 m
   (c) radius = 3.5 cm, height = 7 cm

6. A cylindrical water tank, 7 m in diameter, contains water to a depth of 4 m. Find the total area of the wetted surface. (Use $\pi = \underline{22}$ ).

7. A plot of land is to be fenced using 36 cylindrical posts, each 18 cm in diameter and 2.5 m high. The sides and the top ends of the posts are painted. Find the total surface area painted. (Give your answer to 5 s.f., take $\pi = 3.14$).

8. The total exterior surface area of an open cylinder is 198 $cm^2$ . The exterior diameter is 6 cm. Find the height of the cylinder. (Use $\pi = 3.14$)

9. A closed tank is in the shape of a cylinder of diameter 280 cm and height 210 cm. It is made of galvanised iron sheet. Using the value 22 for $\pi$, find

   (a) the surface area of the tank in $m^2$ .

   (b) the total cost of galvanised iron sheet used to make the tank if it costs K 1 600 per square metre.

## Volume of a cylinder

**Volume** is the amount of space occupied by an object. The SI unit for volume is the **cubic metre ($m^3$ ).**

A cylinder is a solid whose uniform cross-section is a circular surface (Fig. 5.9 ).

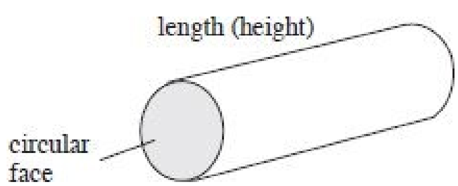



*Fig. 5.9*

Volume of a solid with a uniform cross-section is given by:

Volume = Area of cross-section × length

Thus,

Volume of a cylinder

= Area of the circular face × height

$= \pi r^2 h$

### *Example 5.1*

*Find the volume of a cylindrical container whose diameter is 7.9 cm if its height is 7 cm. Use $\pi$ = 3.14 and give the answer in 3 significant figures.*

### *Solution*

*Volume of cylinder* = $\pi r^2 h$

$= (3.14 \times 3.95^2 \times 7)\ cm^3$

$= 342.9\ cm^3$

$= 343\ cm^3$

### *Example 5.2*

*A cylindrical mug has an inner diameter of 9.2 cm and an inner height of 12.5 cm. Taking π as 3.142, find its capacity.*

### *Solution*

*Volume of cylinder* = $\pi r^2 h$

∴ *Capacity of the mug*

$= (3.142 \times 4.6^2 \times 12.5)\ cm^3$

$= 831.1\ cm^3$ *(to 4 s.f.)*

## Exercise 5.2

1. Find the volume of each of the following cylinders. (Take $\pi = 3.14$)
   (a) Radius 2.1 cm, height 0.3 cm.
   (b) Radius 0.7 cm, height 4.5 cm.
   (c) Radius 0.9 cm, height 14 cm.
   (d) Radius 1.5 m, height 5.6 m.
   (e) Diameter 3.5 m, height 4.9 m.

2. A cylindrical tank has a diameter of 5.0 m and contains 110 000 *l* of water. What is the height of the water in the tank?

   ($1\ m^3 = 1\ 000\ l$ and $\pi = 3.14$)

3. Find the volume of the material of each of the following pipes. (Use $\pi = \frac{22}{7}$ )
   (a) External radius 2.5 cm, thickness 0.5 cm and length 15 cm.
   (b) External diameter 4.4 cm, thickness 0.7 cm and length 24 cm.
   (c) External diameter 15 cm, thickness 0.7 cm and length 3 m.

4. A cylindrical container of diameter 15 cm and depth 20 cm is full of water. If the water is poured into an empty cylindrical jar of diameter 10 cm, find the depth of the water in the jar.

5. A circular pond of diameter 40 m is surrounded by a path 2 m wide.

Calculate the volume of murram required to gravel the path to a depth of 7.5 cm. (Use $\pi = 3.14$)

6. A cylindrical water tank has a diameter of 140 cm. The tank is filled with water. A leak starts at the bottom so that it loses 33 $l$ of water in 1 hour. How long will it take for the water level to fall by 30 cm? (1 $l$ = 1 000 $cm^3$ and $\pi = \frac{22}{7}$).

## Surface area of a pyramid

What solids are formed when the nets in Fig. 5.10 are folded up?

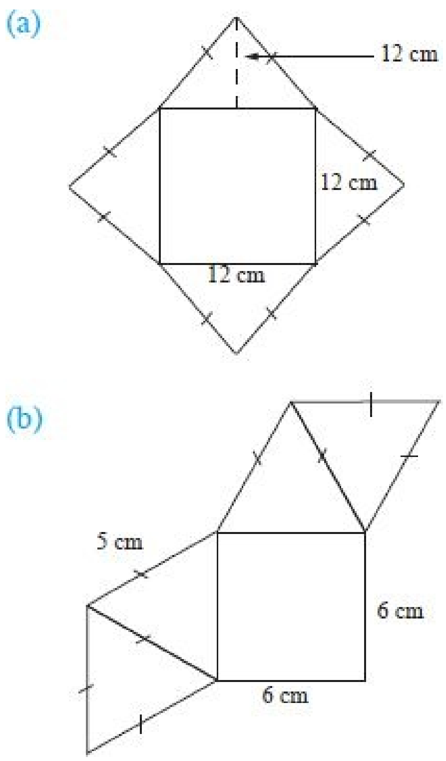



*Fig. 5.10*

Calculate their surface areas.

> The surface area of a pyramid is obtained as the sum of the areas of the slant faces and the base.

Example 5.3 illustrates how to find the surface area of a pyramid.

***Example 5.3***

*Find the surface area of the right pyramid shown in Fig. 5.11 .*

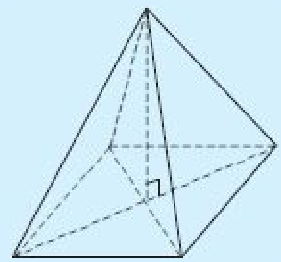


*Fig. 5.11*

## ***Solution***

*Area of the base = 5 × 5 = 25 cm*$^2$

*Each slanting face is an isosceles triangle (Fig. 5.12 ).*

*Its height is* $\sqrt{7^2 - (2.5)^2}$ *cm*

*since* $h^2 = 7^2 - 2.5^2$ .

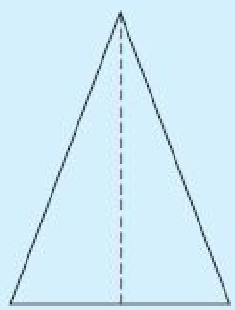


*Fig. 5.12*

*Area of each slanting face*

$$= \frac{1}{2} \times 5 \times \sqrt{7^2 - 2.5^2)}$$

$$= \frac{1}{2} \times 5 \times \sqrt{49 - 6.25)}$$

$$= \frac{1}{2} \times 5 \times \sqrt{42.75}$$

$$= \frac{1}{2} \times 5 \times 6.538$$

$$= 16.345 \text{ cm}^2$$

∴ *Total area of the slanting faces*

$$= 4 \times 16.345 = 65.38 \text{ cm}^2$$

*Total surface area of the pyramid*

$= (25 + 65.38)\ cm^2$

$= 90.4\ cm^2$ *(3 s.f).*

## Exercise 5.3

In Questions 1–7, find the total surface area of the given right pyramid.

1. Height 6 cm; square base, side 9 cm.
2. Height 5 cm; rectangular base, 6 cm by 4 cm.
3. Height 16 cm; triangular base, sides 6 cm, 8 cm and 10 cm.
4. Slant edge 12 cm; rectangular base, 6 cm by 8 cm.
5. Height 10 cm; equilateral triangular base, side 6 cm.
6. Slant edge 4 cm; square base, side 4 cm.
7. Slant height 8 cm; square base, side 5.3 cm.

# 1- 5 REVISION EXERCISES 1

## Revision exercise 1.1

1. In a right-angled triangle, the length of the hypotenuse is $x$ cm, and the lengths of the other two sides are $y$ and $z$ .

   (a) If $z = 4.2$ and $y = 6.8$, find $x$ .
   (b) If $y = 5.73$ and $z = 5.14$, find $x$ .
   (c) If $x = 8.06$ and $y = 5.67$, find $z$.
   (d) If $x = 37.08$ and $z = 23.26$, find $y$ .

2. The sides of a rectangle are 6.75 cm and 5.42 cm long. Find the length of a diagonal of the rectangle.

3. Write down the next three terms in each of the following sequences.

   (a) 3, 7, 11, 15, …
   (b) 42, 35, 28, 21, …
   (c) 6, 12, 24, 48, …

4. A ball falls from a height of 10 m and rebounds each time to a height equal to four fifths of its previous height. What distance will the ball have travelled when it touches the ground for the

   (a) 3rd time?
   (b) 5th time?

5. Factorise and simplify

   (a) $\dfrac{-2x-6}{3x+9}$
   (b) $(3c - d)(m - n) + (3c - d)(2m - 3n)$
   (c) $\dfrac{x^2-6y^2}{2x^2+7xy+3y^2}$

6. Expand and simplify

   (a) $(2n - 7)(3n - 1)$
   (b) $(2x + 1)(2x - 1) + 1$
   (c) $(3x - 2y)(2x + 2y) - 5xy$
   (d) $(2x + 3)(x - 5)$

7. Factorise $xy - 3y + 18$

8. The angles of a triangle are $x$ , $3x$ and $5x$ . Find the value of $x$ .

9. In a kite ABCD, AC is the minor diagonal and BD the major diagonal. Given that DAC = 15º and CBD = 24º, find the size of ∠ACD, ∠ADC and ∠BAD.

10. The base of a right pyramid is a square of side 4 cm. The slant edges are all 6 cm long.
    (a) Draw and label a sketch of the solid.
    (b) Draw a net of the pyramid.
    (c) Use your net to find
        (i) the slant height of the pyramid.
        (ii) the total surface area of the pyramid.

11. A hollow water pipe has a diameter of 1.4 m and length 21 m. Find the surface area of the pipe. (Use $\frac{22}{7}$ ).

12. Calculate the surface area of the solid (Fig. R1.1).

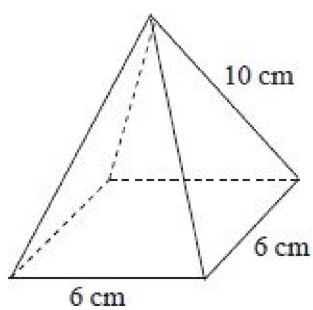



*Fig. R1.1*

## Revision exercise 1.2

1. A rectangular garden has a length of 250 m and a width of 130 m. Find the length of a diagonal path through it.

2. Which of the following triples describes a right-angled triangle?
    (a) (7, 24, 25)
    (b) (8, 15, 17)
    (c) (8, 24, 25)

3. Find a formula for the *n* th term of each of the following sequences.
    (a) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
    (b) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$
    (c) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

(d) $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{7}{10}, \ldots$

4. The third and the fifth terms of an arithmetic sequence are 10 and 16 respectively.

   Determine the first term and the common difference.

5. Factorise and simplify where necessary.

   (a) $c(y - z) - y + z$
   (b) $9 - 9z^2$
   (c) $18 - 8(r + s)^2$
   (d) $\dfrac{x^2 - 9}{5x^2 - 13x - 6}$

6. Simplify $\dfrac{2x - 2}{6x^2 + x + 12} + \dfrac{x - 1}{2x - 3}$

7. Factorise

   (a) $x^2 + x - 20$
   (b) $4x^2 - 9$
   (c) $9x^2 - 12x + 4$

8. If ABCD is an isosceles trapezium in which AD = BC and ∠ADC = 45º, find ∠BCD and ∠ABC.

9. ABCD is a rhombus whose side is 12 cm long. The diagonal BD is 8.2 cm long. If ∠ABD = 70º, find the sizes of ∠BDC, ∠ADC and ∠BAD.

10. The total surface area of a cylinder is 105 $cm^2$. If the radius of the base is 2 cm, use $\pi = 3.14$ to find

    (a) the height of the cylinder.
    (b) the volume of the cylinder.

11. A right pyramid stands on a square base of side 6 cm, and has a height 12 cm. Find

    (a) the slant height of the pyramid.
    (b) the total surface area of the pyramid.

12. Find the total surface area of the solid shown in Fig. R1.2.

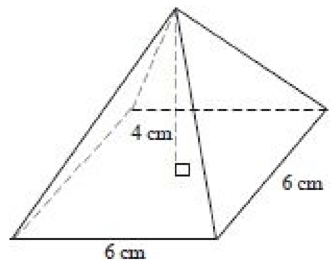



*Fig. R1.2*

## Revision exercise 1.3

1. Complete the following Pythagorean triples.
   (a) (23, …, …)
   (b) (29, …, …)
   (c) (26, …, …)
   (d) (30, …, …)
2. Use the following numbers to generate Pythagorean numbers.
   (a) 3, 5
   (b) 7, 2
   (c) 3, 8
   (d) 5, 6
3. Find the $n$ th term in the following sequences.
   (a) 1, 3, 5, 7, 9 …
   (b) 1, 8, 27, 64 …
4. Write down the first four terms and the tenth term of the sequences whose nth term is defined as
   (a) $3n - 1$
   (b) $n^2 + 1$
   (c) $2^n + 1$
   (d) $2^n - 5n + 1$
5. Factorise
   (a) $x^2 - y^2$ hence evaluate $699^2 - 301^2$
   (b) $(x + 2y)^2 - (x - 2y)^2$
   (c) $3x^2 - 2xy - y^2$
6. Expand and simplify

(a) $5(x + 4) - 3(4x + 2)$
(b) $(4x - 1)(2x - 3)$

7. Simplify the expression
$$\frac{9t^2 - 25a^2}{6t^2 + 19at + 15a^2}$$

8. Find the size of the angles marked by letters $x$ and $y$ in Fig. R1.3 below.

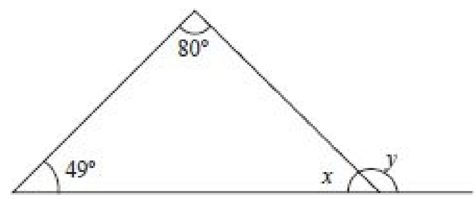



*Fig. R1.3*

9. In trapezium ABCD, AB is parallel to CD. Given that $\angle ABC = 108^\circ$, $\angle ACD = 30^\circ$ and $\angle ADC = 75^\circ$, find $\angle BAD$.

10. Draw the net of each of the solids shown in Fig. R1.4. Use your nets to calculate the total surface area of each solid. (Use $\pi = \frac{22}{7}$ )

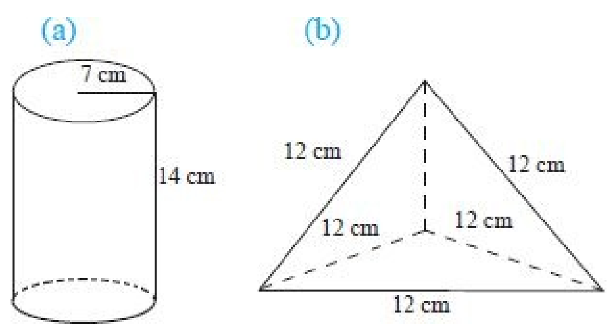



*Fig. R1.4*

11. Find the surface area of the tetrahedron in Fig. R1.5 given that all measurements are in cm.

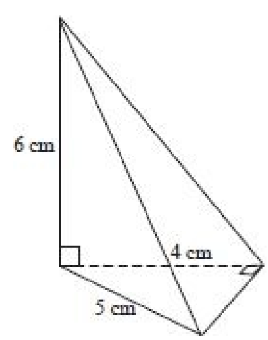


*Fig. R1.5*

12. A closed cylinder whose height is 18 cm has a radius of 3.5 cm. Draw the net of the cylinder and use it to find its total surface area. (Take $\pi = 3.142$)

# 6 LINEAR SIMULTANEOUS EQUATIONS

## An equation in two variables

Suppose Mr. Hussein and Mr. Mwai together have eight children, how can we express this situation in a mathematical statement?

Since there are two numbers that we do not know, it is natural to use two variables (unknowns).

Thus, if Mr. Hussein has x children and Mr. Mwai has y children, together they have (x + y) children.

$x + y = 8$

Table 6.1 shows possible pairs of numbers which make the equation true.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| y | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |

*Table 6.1*

In Table 6.1 , for each value of x, there is a corresponding value of y. We say that such a pair of numbers **satisfies** the equation or it is a **solution** of the equation.

***Example 6.1***

*Show that the following pairs of numbers satisfy the equation $x + 3y = 18$.*

*(a) (0, 6)*

*(b) (3, 5)*

*(c) ($^-3$, 7)*

*(d) (21, $^-1$)*

*In these pairs of numbers the first represents the value of x, while the second represents the value of y.*

***Solution***

*(a) $x + 3y = 18 : (0, 6)$*

$LHS = 0 + 3 \times 6$

$= 0 + 18$

$= 18$

$\therefore LHS = RHS$

*(b) $x + 3y = 18 : (3, 5)$*

$LHS = 3 + 3 \times 5$

$= 3 + 15$

$= 18$

$\therefore LHS = RHS$

*(c) $x + 3y = 18 : (^{-}3, 7)$*

$LHS = {}^{-}3 + 3 \times 7$

$= {}^{-}3 + 21$

$= 18$

$\therefore LHS = RHS$

*(d) $x + 3y = 18 : (21, {}^{-}1)$*

$LHS = 21 + (3 \times {}^{-}1)$

$= 21 + {}^{-}3$

$= 18$

$\therefore LHS = RHS$

*In each case, the left hand side of the equation is equal to the right hand side; all the given pairs of numbers satisfy the equation.*

## Exercise 6.1

In Questions 1 and 2, the given pairs of numbers are such that the first number represents the value of x, while the second represents the value of y.

1. Show that the given pairs of numbers satisfy the equation $y + 2x = 12$.

   (a) (1, 10)
   (b) (5, 2)
   (c) (0, 12)
   (d) $(8, {}^{-}4)$

2. Which of the following pairs of numbers satisfy the equation $3x + 4y = 7$?
   (a) (1, 0)
   (b) (1, 1)
   (c) (5, $^-2$)
   (d) (2, 2)
   (e) (7, 3)
   (f) (9, $^-5$)
3. If x and y represent whole positive numbers, give the first four pairs of numbers which satisfy the equation $3x + y = 15$.
4. Copy and complete Table 6.2 for pairs of numbers that satisfy the equation
   $x + 3y = 17$.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| x | 11 | | | | $^-1$ | 8 | | $^-4$ |
| y | 2 | 0 | $^-2$ | 5 | | | | |

*Table 6.2*

5. If x and y are restricted to whole positive numbers, give six pairs of numbers that satisfy, each of the following equations.
   (a) $3x - y = 8$
   (b) $x - 2y = 1$

   Is there any pair of numbers that satisfies both equations? If yes, state the pair.

## Pairs of equations in two variables

Consider the equations $3x + 2y = 12$

$$4x - 2y = 2$$

Each equation contains unknown quantities x and y. The solutions of the equations are the values of x and y which satisfy both equations simultaneously. Equations such as these are called **simultaneous equations** .

To solve simultaneous equations, we look for a pair of numbers which satisfies the two equations at the same time.

## Solution of simultaneous equations by elimination method

### *Example 6.2*

*Solve the simultaneous equations*

$3x + 2y = 12$ ………… (1)

$4x - 2y = 2$ ………… (2)

### *Solution*

*Since the LHS of an equation is always equal to the RHS, then $(3x + 2y)$ and 12 in equation (1) can be used interchangeably.*

*Also, $(4x - 2y)$ and 2 in equation (2) represent the same quantity.*

*Therefore, we can combine the given equations thus:*

$$(3x + 2y) + (4x - 2y) = 12 + 2$$

$$3x + 2y + 4x - 2y = 14$$

$$7x = 14$$

$$x = 2$$

*In any of the original equations, we can use 2 instead of x, i.e. we can substitute 2 for x.*

*Using equation (1),*

$3x + 2y = 12$ *becomes* $3 \times 2 + 2y = 12$

$$6 + 2y = 12$$

$$2y = 6$$

$$y = 3$$

*The solutions of the simultaneous equations are therefore $x = 2$ and $y = 3$*

**Note:** When we added equation (1) to equation (2), the variable y was eliminated. We remained with a simple equation in one variable x.

### *Example 6.3*

*Solve the simultaneous equations*

$2x + 4y = {}^{-}12$ ……… (1)

$5x + 4y = {}^{-}33$ ……… (2)

***Solution***

*If we subtract equation (2) from equation (1) we get a simple equation in one unknown.*

*Thus,* $(2x + 4y) - (5x + 4y) = {}^{-}12 - ({}^{-}33)$

$$2x + 4y - 5x - 4y = {}^{-}12 + 33$$

$$^{-}3x = 21$$

$$x = {}^{-}7$$

*To make the given equations true, x must be equal to* $^{-}7$*.*

*Now, in equation (1) or (2) we use* $^{-}7$ *instead of x to solve for y.*

*Thus,*

$2x + 4y = {}^{-}12$ *becomes* $(2 \times {}^{-}7) + 4y = {}^{-}12$

*i.e.* $^{-}14 + 4y = {}^{-}12$

$$4y = 2$$

$$y = \frac{1}{2}$$

*Use* $^{-}7$ *for x in equation (2) and confirm that* $y = \frac{1}{2}$ *.*

In Examples 6.2 and 6.3 we were able to get rid of one of the variables by addition or subtraction, because the coefficients of one of the unknowns in both equations were either the same or opposite. This method of getting rid of one of the variables by addition or subtraction is known as the **elimination method** .

## Exercise 6.2

In this exercise, examine each of the pairs of equations carefully, and decide when to add and when to subtract. Then solve the simultaneous equations.

1. $3x - y = 8$
   $x + y = 4$
2. $5x - y = 18$
   $3x + y = 14$
3. $3x - 2y = 0$
   $x - 2y = {}^{-}4$
4. $4x - 3y = 16$
   $2x + 3y = 26$

5. $x + 2y = 11$
   $x - 2y = 3$
6. $3x + 2y = 12$
   $4x + 2y = 2$
7. $3a - 2b = 11$
   $2a - 2b = 10$
8. $r - s = 1$
   $^{-}r - s = 13$
9. $2m - n = 11$
   $3m + n = 49$
10. $5x + 3y = 9$
    $^{-}5x + 2y = 1$
11. $7x - 2y = 29$
    $7x + y = 38$
12. $10x - 3y = 36$
    $x + 3y = 18$
13. $5x - 6y = 16$
    $7x + 6y = 44$
14. $x + 6y = {}^{-}5$
    $x - 9y = 0$
15. $6x + 4y = 24$
    $7x - 4y = 2$
16. $5x + 3y = 77$
    $15x - 3y = 3$

## More pairs of simultaneous equations

Consider the equations

$3x - 2y = 8$

$x + 5y = {}^{-}3$

In this pair of equations we cannot eliminate any variable by simple addition or

subtraction.
We must first make the coefficient of x or y the same in both cases.

***Example 6.4***

*Use elimination method to solve the simultaneous equations*

$3x - 2y = 8$

$x + 5y = {}^{-}3$

## *Solution*

$3x - 2y = 8$ ………… (1)

$x + 5y = {}^{-}3$ ………… (2)

*Leave (1) as it is:* $3x - 2y = 8$ …… (1)

*Multiply (2) by 3:* $\underline{3x + 15y = {}^{-}9}$ …….(3)

*Subtract (3) from (1)* ${}^{-}17y = 17$

$y = {}^{-}1$

*Use* ${}^{-}1$ *instead of y in (1)*

$3x - 2({}^{-}1) = 8$

$3x + 2 = 8$

$3x = 6$

$x = 2$

*Check in (2):*

LHS $= 2 + 5({}^{-}1) = 2 - 5 = {}^{-}3$, RHS $= {}^{-}3$

***Example 6.5***

*Solve the simultaneous equations*

$3x + 4y = 10$ ………… (1)

$2x - 3y = 1$ ………… (2)

## *Solution*

*Multiply (1) by 2:* $6x + 8y = 20$ ………… (3)

*Multiply (2) by 3:* $\underline{6x - 9y = 3}$…………(4)

*Subtract (4) from (3):* $17y = 17$

$y = 1$

*Substitute $y = 1$ in : $3x + (4 \times 1) = 10$*

$$3x + 4 = 10$$
$$3x = 6$$
$$x = 2$$

*Check in (2):*

*LHS = $(2 \times 2) - (3 \times 1) = 4 - 3 = 1$, RHS = 1*

## *Example 6.6*

*Solve the simultaneous equations*

$2x + 7y = 15$

$5x - 3y = 19$

## *Solution*

$2x + 7y = 15$ ……… (1)

$5x - 3y = 19$ ……… (2)

*Eliminate x:*

(1) × 5: $10x + 35y = 75$ ……… (3)

(2) × 2: $10x - 6y = 38$ ……….(4)

(3) – (4): $41y = 37$

$$y = \frac{37}{41}$$

*Note that since y is a fraction, substituting $y = \frac{37}{41}$ in equation (1) or (2) would make work more difficult. We obtain x by eliminating y (as was done for x) i.e.*

(1) × 3: $6x + 21y = 45$

(2) × 7: $35x - 21y = 133$

(4) + (5): $41x = 178$

$$\therefore x = \frac{178}{41}$$

*Hence, the solution is $x = 4\frac{14}{41}$, $y = \frac{37}{41}$*

*Note that this is called the method of **eliminating twice.***

**To solve simultaneous equations by elimination method:**

1. Decide which variable to eliminate.
2. Make the coefficients of the variable the same in both equations.

3. Eliminate the variable by addition or subtraction as is appropriate.
4. Solve for the remaining variable.
5. Substitute your value from 4 above in any of the original equations to solve for the other variable.

As with simple equations, x and y can have any value. They can be positive or negative, whole numbers or fractions.

## Exercise 6.3

Solve the simultaneous equations.

1. $3x + 2y = 16$
   $2x - y = 6$
2. $2x - y = 7$
   $5x - 3y = 16$
3. $2x + 3y = 27$
   $3x + 2y = 13$
4. $3x + 2y = 13$
   $2x + 3y = 12$
5. $x = 5 - 2y$
   $5x + 2y = 1$
6. $x + y = 0$
   $2y - 3x = 10$
7. $3x + y = 12$
   $2x - 3y = 8$
8. $4y - x = 7$
   $3y + 4x = {}^{-}9$
9. $5n + 2m = 10$
   $3m + 7n = 29$
10. $2x - 4y = 8$
    $3x - 2y = 8$

11. $2x + 3y = 600$
$x + 2y = 350$

12. $6a - b = {}^{-}1$
$4a + 2b = {}^{-}6$

13. $w - 2z = 5$
$2w + z = 5$

14. $2x - 4y = {}^{-}10$
$3x + y - 6 = 0$

15. $9x + 3y = 4$
$3x - 6y = {}^{-}1$

16. $2x - y = 6$
$3x + 2y = 18$

17. $3x - 4y = {}^{-}5$
$2x + y = 6$

18. $2x - 7y = {}^{-}10$
$9y + 5x = 6$

## Solution of simultaneous equations by substitution method

Consider the equations

$3x - 5y = 23$ …………. (1)

$x - 4y = 3$ ………… (2)

Using equation (2), add 4y to both sides:

$x - 4y + 4y = 3 + 4y$

$x = 3 + 4y$ ………… (3)

In equation (3), x is said to be expressed or solved in terms of y.

In equation (1), use (3 + 4y) in place of x:

$3x - 5y = 23$ becomes $3(3 + 4y) - 5y = 23$

$9 + 12y - 5y = 23$

$9 + 7y = 23$

$7y = 14$

$y = 2$

Substitute $y = 2$ in equation (3):

$$x = 3 + 4(2)$$
$$= 3 + 8$$
$$= 11$$

Use equation (1) to check that the solutions are correct. This method of solving simultaneous equations is called **substitution method** .

### *Example 6.7*

*Solve the simultaneous equations*

$$\frac{x+y}{3} - \frac{x-y}{4} = \frac{2}{3} \text{ ………… (1)}$$

$$\frac{2x-3}{3} - \frac{2y+3}{4} = \frac{^{-}19}{12} \text{ ………… (2)}$$

### *Solution*

*Multiply (1) by 12 to remove the denominators.*

$$4x + 4y - 3x + 3y = 8$$
$$x + 7y = 8 \text{ ………… (3)}$$

*Multiply (2) by 12 to remove the denominators.*

$$8x - 12 - 6y - 9 = {}^{-}19$$
$$8x - 6y = 2 \text{ ………… (4)}$$

*Using (3), express x in terms of y:*

$$x = 8 - 7y \text{ ………… (5)}$$

*Substitute (5) in (4) to eliminate x:*

$$8(8 - 7y) - 6y = 2$$
$$64 - 56y - 6y = 2$$
$$64 - 62y = 2$$
$$62y = 62$$
$$y = 1$$

*Substitute y = 1 in (5):*

$$x = 8 - 7(1)$$
$$= 8 - 7$$
$$= 1$$

*Substitute y = 1 and x = 1 in (1) or (2) to check that the solutions are correct.*

**To solve simultaneous equations by substitution method:**

1. First decide which variable is easier to eliminate.
2. Using the simpler of the two equations, express the variable to be eliminated in terms of the other.
3. Using the other equation, substitute the equivalent for the variable to be removed.
4. Solve for the remaining variable.
5. By substitution solve for the other unknown.
6. By substitution, check whether your solutions satisfy the equations.

## Exercise 6.4

In Questions 1 to 4, find y in terms of x.

1. $4x - y = 12$
2. $2x + 5y = 10$
3. $\frac{1}{3}x - 4y = 16$
4. $3x - \frac{1}{4}y = 10$

In Questions 4 to 6, express x in terms of y.

5. $x - 5y = 3$
6. $9x + 4y = 0$
7. $\frac{1}{3}x = \frac{1}{6}(y - 1$
8. $\frac{1}{3}(x - 2) - y = 2$

Use substitution method to solve the following pairs of simultaneous equations.

9. $a + b = 3$

   $4a - 3b = 5$
10. $w - 2z = 5$

$2w + z = 5$

11. $x + y = 0$

$2y - 3x = 10$

12. $x = 5 - 2y$

$5x + 2y = 1$

13. $4x - 3y = 1$

$x - 4 = 2y$

14. $6a - b = {}^{-}1$

$4a + 2b = {}^{-}6$

15. $4m - n = -3$

$8m + 3n = 4$

16. $5q + 2p = 10$

$3p + 7q = 29$

17. $\frac{1}{v} + \frac{1}{u} = \frac{1}{5}$

$\frac{1}{v} - \frac{1}{u} = \frac{2}{5}$

18. $2s - 4t = 8$

$3s - 2t = 8$

19. $2x - 4y + 10 = 0$

$3x + y - 6 = 0$

20. $\frac{1}{2}a - 2b = 5$ $a - 2b = 5$

$\frac{1}{2}a + b = 1$ $a + b = 1$

21. $\frac{2y}{5} + \frac{z}{3} = 2\frac{2}{3}$

$y = 2(z + 1)$

22. $\frac{a-1}{2} + \frac{b+1}{5} = \frac{1}{5}$

$\frac{a+b}{3} = b - 1$

## Graphical solution of simultaneous linear equations

Consider the pair of simultaneous equations y + 2x = 6

x – y = 3

Table 6.3 (a) gives some of the ordered pairs of values that satisfy the equation y + 2x = 6.

Table 6.3 (b) gives pairs of values for the equation x – y = 3.

(a)

| x | 0 | 1 | 2 | 3 | 4 | 5 | . . . |
|---|---|---|---|---|---|---|---|
| y | 6 | 4 | 2 | 0 | −2 | −4 | . . . |

(b)

| x | 0 | 1 | 2 | 3 | 4 | 5 | . . . |
|---|---|---|---|---|---|---|---|
| y | −3 | −2 | −1 | 0 | 1 | 2 | . . . |

*Table 6.3*

The ordered pair x = 3, y = 0, i.e. (3, 0) appears in both tables; and it is the only pair that does so. It is the only pair of values that satisfies both equations simultaneously (i.e. at the same time). Hence, the solution of the simultaneous equations is x = 3, y = 0.

The result can be obtained by drawing the graphs of the two lines as in Fig. 6.1 . The two lines intersect at the point (3, 0). This is the only point that is on both lines.

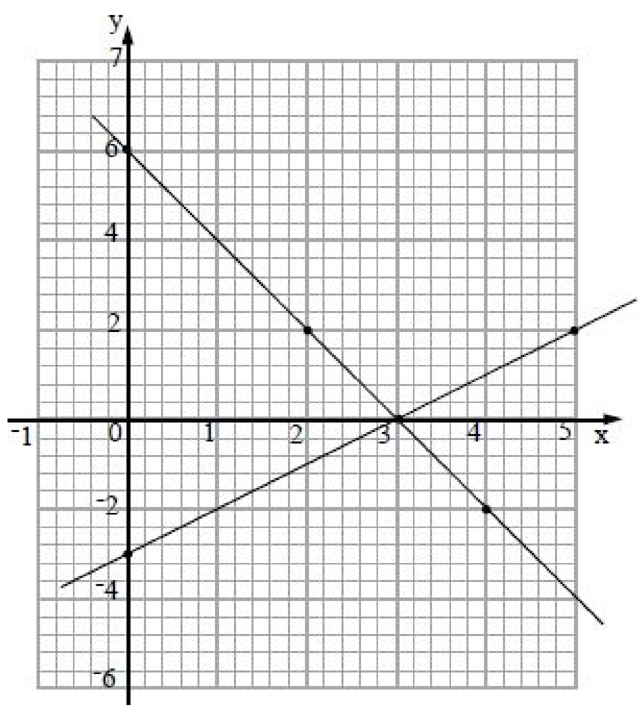



*Fig. 6.1*

The coordinates of the point at which the lines intersect give the solution of the simultaneous equations.

### *Example 6.8*

*Solve graphically the simultaneous equations*

$$x - 2y = {}^{-}1$$

$$2x - y = 4$$

### *Solution*

***Step 1:***

*Make a table of values for each equation. Three pairs of values are sufficient for each (Table 6.4 (a) and (b)).*

*(a)* $x - 2y = {}^{-}1$

*(b)* $2x - y = 4$

(a)

| | | | |
|---|---|---|---|
| $x$ | 0 | 1 | 3 |
| $y$ | $\frac{1}{2}$ | 1 | 2 |

(b)

| | | | |
|---|---|---|---|
| $x$ | 0 | 1 | 2 |
| $y$ | $^{-}4$ | $^{-}2$ | 0 |

*Table 6.4*

***Step 2:***

*Choose a suitable scale and plot the points. Draw the lines. Extend if necessary, so that they intersect. Fig. 6.2 shows the graphs of the two lines.*

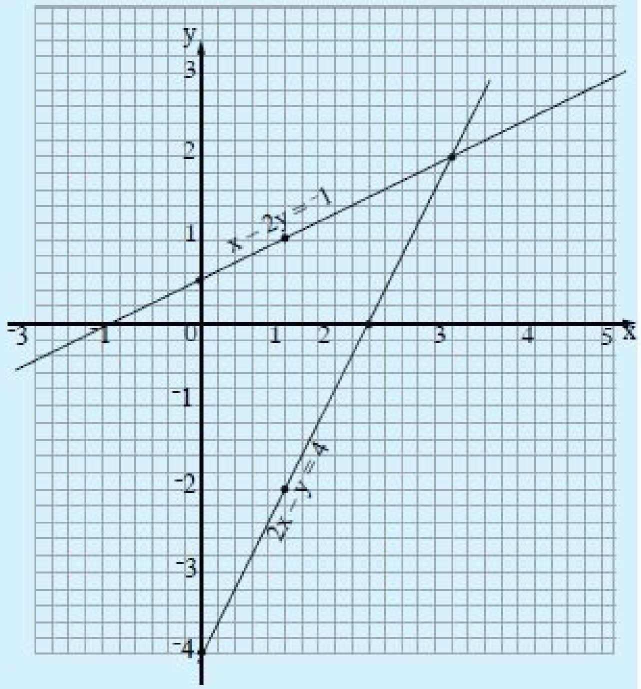



*Fig. 6.2*

***Step 3:***

*Read the coordinates of the point of intersection.*

*From the graph, the lines intersect at (3, 2). the solution of the simultaneous equations is*

*∴ $x = 3, y = 2$*

## Exercise 6.5

Solve graphically the following pairs of simultaneous equations.

1. $x + y = 3$
   $y = 3x - 1$
2. $2x = 3 + y$
   $7x + 2y = 16$
3. $2x - y = 3$
   $2y = -x + 14$
4. $3x + 2y = 0$
   $5x + y = 7$

5. $y + 1 = 2x$
   $2y + x + 7 = 0$
6. $3y - x = 4$
   $2x - 5y = {}^{-}7$
7. $3x + 4y = 3.5$
   $7x - 6y = 0.5$
8. $2y + 3x = {}^{-}5$
   $3y + 2 = x$
9. $4x - y = 2$
   $6x + 4y = 25$
10. $4x - 2y = 4$
    $2x - 3y = 0$
11. $2x - y = {}^{-}1$
    $x - 2y = 4$
12. $12x + 6y = 12$
    $2x - 3y = {}^{-}2$
13. $10x - l0y = 3$
    $2x + 3y = 3.1$
14. $x - y = {}^{-}1$
    $4x - 8y = 4$
15. $2y + 2 = 3x - 6$
    $\frac{y-1}{2} + \frac{x-3}{2} = \frac{x+3}{3}$
16. $5x - 2y = 1$
    $4x + 3y = {}^{-}10.7$

## Forming and solving simultaneous equations

Consider the following situation:

Two years ago, a man was seven times as old as his son. In three years time, he will be only four times as old as his son. Find their present ages.

If the present age of the man is $m$ and the present age of the son is $s$ years, then two years ago, the man's age was $(m – 2)$ years, and the son's age was $(s – 2)$ years.

$\therefore m – 2 = 7(s – 2)$ ………… (1)

In three years time, the man's age will be

$(m + 3)$ years.

In three years time, the son's age will be $(s + 3)$ years.

$\therefore m + 3 = 4(s + 3)$ ………… (2)

Equations (1) and (2) can be written simply as

$m = 7s – 12$ ………… (3)

$m = 4s + 9$ ………… (4)

Since the LHS are equal, the RHS are also equal.

$\therefore 7s – 12 = 4s + 9$

$3s = 21$

$s = 7$

Substituting 7 for $s$ in (3):

$m = 7 \times 7 – 12$

$= 49 – 12$

$= 37$

The present age of the man is 37 years.

The present age of the son is 7 years.

Thus, we have formed and solved simultaneous equations from the given situation.

## *Example 6.9*

*A two digit number is such that its value equals four times the sum of its digits. If 27 is added to the number, the result is equal to the value of the number obtained when the digits are interchanged. What is the number?*

## *Solution*

*Let the tens digit be $x$.*

*Let the ones digit be $y$.*

*$\therefore$ the value of the number is $10x + y$ and the sum of the digits is $x + y$.*

$10x + y = 4(x + y)$

$10x + y = 4x + 4y$

$6x = 3y$

$2x = y$

$2x - y = 0$ …………… (1)

*The value of the number formed by interchanging the digits is* $10y + x$.

$\therefore 10x + y + 27 = 10y + x$.

$9x - 9y + 27 = 0$

$9x - 9y = {}^{-}27$

$x - y = {}^{-}3$ …………… (2)

*Subtract (2) from (1):*

$$\begin{array}{rl} 2x - y &= 0 \\ x - y &= {}^{-}3 \\ \hline x &= 3 \end{array}$$

*Substitute* $x = 3$ *in (1):*

$(2 \times 3) - y = 0$

$6 - y = 0$

$\therefore y = 6$

*The original number is 36.*

*Check by using the information in the question.*

## Exercise 6.6

1. The sum of two numbers is 10, and their difference is 6. Make a pair of equations and solve them simultaneously to find the numbers.
2. Mary is one year older than June, and their ages add up to 15. Form a pair of equations and solve them to find the ages of the girls.
3. Two books have a total of 500 pages. One book has 350 pages more than the other. Find the number of pages in each book.
4. A bag contains K 5 coins and K 10 coins. There are 14 coins in all, and their value is K 105. Find the number of each type of coin.
5. Two numbers are such that twice the larger number differs from thrice the smaller number by four. The sum of the two numbers is 17. Find the numbers.

6. If 5 is added to both the numerator and denominator of a fraction, the result is $\frac{4}{7}$ . If 1 is subtracted from both the numerator and denominator, the result is $\frac{2}{5}$ . Find the fraction.
7. The cost of 3 sheep and 2 goats is K 14 400. If 4 sheep and a goat cost K 15 200, find the cost of two goats and a sheep.
8. A wire 200 cm long is bent to form a rectangle. The length of the rectangle is 3 cm longer than the width. Find the dimensions of the rectangle.
9. A man is 22 years older than his son, and their total age is 48 years. Form a pair of equations and solve them to find the ages of the man and his son.
10. The length of a rectangle is 2 m more than its width, and the perimeter is 8 m. Find the length and breadth of the rectangle.
11. The sum of the number of edges and faces of a solid is 20. The difference between the number of edges and faces is 4. Find the number of edges and faces.
12. The velocity in km/h of a car after t hours is given by the formula $v = u + at$, where u and a are constants. Given that $v = 50$ when $t = 2$ and $v = 140$ when $t = 5$, find

    (a) the constants u and a.
    (b) the velocity when $t = 7$ hours.
    (c) the time at which $v = 260$ km/h.
13. The sum of the digits in a three digit number is nine. The tens digit is half the sum of the other two and the hundreds digit is half the units digit. Find the number.
14. Asale and Mbiya each collected a number of stones to use in an arithmetic lesson. If Asale gave Mbiya 5 stones, Mbiya would have twice as many as Asale. If Asale had five stones less than Mbiya, how many stones did each have?

# 7 PROPORTIONS

## Ratio

In order to discuss proportions, we will need to define the term ratio.

Two quantities of the **same kind** e.g. lengths of line segments AB and CD, 8 cm and 6 cm respectively, may be compared by division i.e. giving one quantity as a fraction of another:

$\frac{AB}{CD} = \frac{4}{3}$ which means that

$AB = \frac{4}{3} \times CD$

Comparison by division is called **ratio** .

We write the ratio $\frac{AB}{CD} = \frac{4}{3}$ as AB : CD

= 4 : 3.

This is often read as 'the length AB to the length CD is equal to four to three'.

The **units** of the two quantities may be different, e.g. 45 min 30 s : 2 h, but the quantities are the same kind, and hours, minutes or seconds may each be expressed in terms of the other two.

### *Example 7.1*

*Susie has 4 sweaters and 5 blouses. What is the ratio of the number of (a) sweaters to blouses?*

*(b) blouses to sweaters?*

### *Solution*

*(a) The ratio of the number of sweaters to the number of blouses is 4 : 5.*

*(b) The ratio of the number of blouses to the number of sweaters is 5 : 4.*

## Direct and inverse proportion

## Direct proportion

Two quantities are said to be related in direct proportion if they increase (or decrease) in the same ratio.

For example, if a man walks steadily at a speed of 4 km/h, the distance he walks is directly proportional to the time he takes.

In 1 h, he walks 4 km; in 2 h (twice the time), he walks 8 km (twice the distance); in 4 h (4 times the time), he walks 16 km (4 times the distance), and so on.

Which of the following pairs of quantities are directly proportional? Why are the others not?

Discuss

(a) Distance travelled at a constant speed; time taken.

(b) Age of child; height of child.

(c) Number of exercise books in a pile; height of pile.

(d) Number of men; time taken to do a job.

(e) Number of similar pens; total cost of the pens.

### *Example 7.2*

*Pat drives 63 km in 47 min. How far will she drive in 2 h 21 min at the same speed?*

### *Solution*

*Distance is directly proportional to time taken.*

*New time taken is 2 h 21 min = 141 min*

∴ *time increases in the ratio 141 : 47 = 3 : 1*

∴ *distance increases to*

$$\frac{141}{47} \times 63 \text{ km} = \frac{3}{1} \times 63 \text{ km}$$

$$= 189 \text{ km}$$

*i.e. Pat drives 189 km in 2 h 21 min*

### *Example 7.3*

*A firm pays K 19 600 every week in wages. If the number of hours each employee works is decreased from 40 to 38 and the hourly rate of pay increased by K 10 from K 70, find the new weekly total wages.*

### *Solution*

*Number of hours worked decreases in the ratio 38:40.*

*∴ Total wages decrease in the ratio 38:40.*

*Hourly rate of pay increases in the ratio 80:70.*

*∴ Total wages increase in the ratio 80:70.*

*Hence, the new weekly total wages*

$$= \frac{38}{40} \times \frac{80}{70} \times K\ 19\ 600$$

$$= K\ 21\ 280$$

### *Example 7.4*

*x and y are directly proportional. Fill in the missing values in Table 7.1 .*

| | | | | | | |
|---|---|---|---|---|---|---|
| *x* | 1 | 2 | 3 | 4 | 5 | 6 |
| *y* | – | – | 90 | – | – | – |

*Table 7.1*

### *Solution*

*If the missing values, from left to right, are a, b, c, d and e;*

*1 : 2 = a : b, 2 : 3 = b : 90, 3 : 4 = 90 : c, 4 : 5 = c : d, and 5 : 6 = d : e*

*Thus,* $\frac{2}{3} = \frac{b}{90}$ $\therefore\ b = \frac{2}{3} \times 90 = 60$

$\frac{1}{2} = \frac{a}{60}$ $\therefore\ a = \frac{1}{2} \times 60 = 30$

$\frac{3}{4} = \frac{90}{c}$ $\therefore\ c = \frac{4}{3} \times 90 = 120$

$\frac{4}{5} = \frac{120}{d}$ $\therefore\ d = \frac{5}{4} \times 120 = 150$

$\frac{5}{6} = \frac{150}{e}$ $\therefore\ e = \frac{6}{5} \times 150 = 180$

*Thus, the complete table is Table 7.2 .*

| | | | | | | |
|---|---|---|---|---|---|---|
| *x* | 1 | 2 | 3 | 4 | 5 | 6 |
| *y* | 30 | 60 | 90 | 120 | 150 | 180 |

*Table 7.2*

What do you notice about the values of $\frac{x}{y}$ in Table 7.2 ?

From Example 7.4, we notice that:

If quantities $x$ and $y$ are directly proportional, $\frac{x}{y}$ is a constant, and $x$ *and* $y$ increase or decrease in the same ratio.

## Exercise 7.1

1. If a bus journey of 120 km costs K 500, how much should a journey of 150 km cost assuming that the fare is charged in proportion to distance?
2. Six tablets of a certain type of soap cost K 440. What is the cost of 11 tablets?
3. Twelve men weed 3 ha of land in 5 days. How many hectares of land do 18 men weed in the same number of days assuming that they work at the same rate?
4. If n is directly proportional to m, fill in the missing values in Table 7.3 .

| | | | | | | |
|---|---|---|---|---|---|---|
| m | 8 | 18 | 24 | 48 | 60 | — |
| n | — | — | 42 | — | — | 119 |

*Table 7.3*

5. A speed of 54 km/h is equivalent to 15 m/s. Find the equivalent of
   (a) 36 km/h in m/s
   (b) 210 m/s in km/h
6. Twelve girls earn K 1 800 in 6 days. How much will 8 girls earn in 9 days?
7. If a railway corporation charges K 2 700 to carry 36 tonnes of load for 28 km, how much would it charge to carry 42 tonnes for 54 km?

## Graphical representation of direct proportion

We say that distance is **directly proportional** to the time taken, or distance **varies directly** as time.

The symbol used for ‘varies as’ is $\propto$.

Thus, if distance is D km and time is T hours, then D varies directly as T, i.e. D $\propto$ T.

This means that $\frac{D}{T}$ is a constant.

In general, if M $\propto$ L, then L $\propto$ M and $\frac{M}{L}$ = k and $\frac{L}{M}$ = c, where k and c are constants of proportionality.

***Example 7.5***

*Given that A $\propto$ B and that A = 1 $\frac{7}{8}$ when B = $\frac{5}{6}$, find a relationship between A and B. Hence find A when B = 0.4 and B when A = 7.5.*

***Solution***

*A $\propto$ B. Therefore, $\frac{A}{B}$ is a constant Let this constant be k.*

*Thus, $\frac{A}{B}$ = k Therefore, A = Bk.*

$1\frac{7}{8} = \frac{5}{6}k$ *(substituting given values of A and B)*

$\therefore k = \frac{15}{8} \times \frac{6}{5}$

$k = \frac{9}{4}$ *(This is the constant of proportionality)*

$\therefore A = \frac{9}{4}B$. *(This is the required law connecting A and B)*

*When B = 0.4,*

$A = \frac{9}{4} \times 0.4$ *(since A = k × B)*

$= 0.9$

*When A = 7.5*

$7.5 = \frac{9}{4}B$

$B = \frac{7.5 \times 4}{9}$

$= 3\frac{1}{3}$

Plot the points $(1\frac{7}{8}, \frac{5}{6})$, (0.9, 0.4) and $(7\frac{1}{2}, 3\frac{1}{3})$, and join them using a straight line. What do you notice about the gradient of the line? Which special point does the line pass through?

You should have noticed that:

If the values of A are plotted against the corresponding values of B, the result will be a straight line through the origin. The gradient of the line will be equal to the constant of proportionality.

In general,

If two variables $x$ and $y$ are directly proportional, then $y = kx$ where $k$ is a constant. Since $y = kx$ is a linear relation, a graph of $y$ against $x$ would be a straight line through the origin, gradient $k$ (Fig. 7.1 ).

Conversely, if the graph of $y$ against $x$ is linear through the origin, then $y$ varies directly as $x$ , and the gradient of the line represents the constant of proportionality.

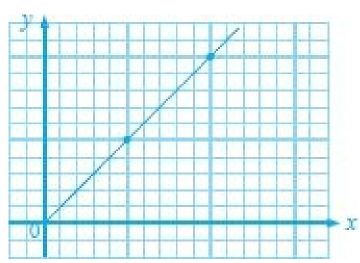



*Fig. 7.1*

## Exercise 7.2

1. If A ∝ B and A = 140 when B = 35, find the law of propotionality and the value of B when A = 176.
2. If $x \propto y$ and $x = 15$ when $y = 18$, find $x$ when $y = 15$ and $y$ when $x = 21$.
3. Table 7.4 shows corresponding values of two variables $x$ and $y$ . Show graphically that $y \propto x$ . Find the law of proportionality in the form $y = kx$ .

| $x$ | 3 | 4 | 6 | 9 | 12 |
|---|---|---|---|---|---|
| $y$ | 2.6 | 3.5 | 5.2 | 7.8 | 10.4 |

*Table 7.4*

Use your graph to read off $x$ when $y = 7$, and $y$ when $x = 3.8$.

4. The quantity, Q, of a certain commodity, in kilograms, and the corresponding cost, P, in Kwacha are given in Table 7.5 .

| Q | 2 | 4 | 5 | 7 |
|---|---|---|---|---|
| P | 80 | 160 | 200 | 280 |

*Table 7.5*

Show graphically that Q ∝ P. Find the law connecting Q and P. Read off the value of Q when P = 450.

5. Table 7.6 shows the corresponding values of the average diameter of an oak tree at different ages.

| Age in years | 10 | 20 | 30 | 50 | 70 | 100 | 150 |
|---|---|---|---|---|---|---|---|
| Diameter in cm | 12 | 24 | 35 | 58 | 81 | 254 | 381 |

*Table 7.6*

Using a graph, estimate

(a) the diameter for an age of 40 years.

(b) the age of which the diameter is 96 cm.

## Non-linear direct proportionality

Direct proportionality does not always give rise to a linear relation.

Consider Table 7.7 .

| A | 20 | 45 | 80 | 125 | 180 | 245 | 320 |
|---|---|---|---|---|---|---|---|
| $h$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

*Table 7.7*

If $A \propto h$ , then the ratio $\frac{A}{h}$ should be a constant. But in this case, it is not!

∴ A and $h$ are not **directly proportional** .

Now copy and complete Table 7.8 .

| A | 20 | 45 | 80 | 125 | 180 | 245 | 320 |
|---|---|---|---|---|---|---|---|
| $h$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $h^2$ | 4 | 9 | | | | | |

*Table 7.8*

From Table 7.8 , you should notice that A varies directly as the square of $h$, i.e. $\frac{A}{h^2} = 5.$

$\therefore$ A = $5h^2$

We can also say that $h^2$ varies directly as A.

Thus $h^2 \propto A$

i.e. $h^2 = kA$

Thus, 4 = $20k$ (from 1st pair of values in Table 7.8 )

$\therefore \quad k = \frac{4}{20} = \frac{1}{5}.$

In this case, the constant of proportionality is $\frac{1}{5}$ .

**Note:** If $h^2$ varies directly as A, then $h$ varies directly as the square root of A.

i.e $h^2 \propto A$

$\Rightarrow h^2 = kA$

$\Rightarrow h = \pm \sqrt{kA}$

$= \pm \sqrt{k} \cdot \sqrt{A}$

Since $k$ is a constant, $\pm \sqrt{k}$ must also be a constant, say $c$.

$\therefore \quad h = c\sqrt{A}$

This means $h \propto \sqrt{A}$

By graphing A against $h^2$ , verify that A and $h^2$ are directly proportional.

Note also that: If $A \propto h^2$ , the result of multiplying $h$ by 2 means multiplying A by $2^2$ .

In general, if $y \propto x^n$ , $\frac{y}{x^n} = k \Rightarrow y = kx^n$ where $k$ is a constant. Plotting $y$ against $x^n$ gives a straight line through the origin, gradient $k$.

## Exercise 7.3

1. If $y = 3$ when $x = 5$, find the relationship between $x$ and $y$ if
   (a) $y \propto x$
   (b) $y \propto x^3$
   (c) $y \propto \sqrt{x}$

   In each case, find the value of $y$ when $x = 7$.

2. Copy and complete the following:

(a) If $A \propto r^2$, $r \propto$ __

(b) If $V \propto r^3$, $r \propto$ __

(c) If $x \propto \sqrt{y}$, $y \propto$ __

3. If $y \propto \sqrt{x}$ and $y = 0.4$ when $x = 4$,

(a) find $y$ in terms of $x$.

(b) find $y$ when $x = 100$ and $x$ when $y = 1.4$.

4. If $y$ varies directly as the cube of $x$, and $y = 50$ when $x = 5$,

(a) find $y$ in terms of $x$.

(b) find $y$ when $x = 1$ and $x$ when $y = 6\frac{1}{4}$ .

## Inverse proportion

When one of two quantities **increases (decreases)** in the ratio $\frac{a}{b}$ and the other **decreases (increases)** in the ratio $\frac{b}{a}$ , the two quantities are related in **inverse proportion** .

For example, a person travels the same distance at different speeds. If the distance can be covered in 4 h at a speed of 40 km/h, then it can be covered in 2 h (half as long) at 80 km/h (twice as fast); in 1 h ($\frac{1}{4}$ of the time) at 160 km/h (4 times as fast); and so on.

Time taken is inversely proportional to the speed of travel.

Which of the following pairs of quantities are inversely proportional?

(a) Time taken to travel a fixed distance; speed.

(b) Distance travelled by a car; amount of petrol used.

(c) Number of men available; time taken to do a job.

(d) Number of units of an article bought for a fixed amount of money; cost per unit of article.

(e) Number of articles bought for a fixed amount of money; cost per article.

## *Example 7.6*

*Temwa has enough money to buy 65 kg of maize at K 30 per kg. How much can she buy if the price is reduced to K 25 per kg?*

## *Solution*

*Amount bought is inversely proportional to price.*

*Price decreases in the ratio 25 : 30 = 5 : 6.*

*∴ amount bought increases in the ratio 30 : 25 = 6 : 5*

*i.e. amount bought increases to* $\frac{6}{5} \times 65$ *kg*

*= 78 kg*

## *Example 7.7*

*Table 7.9 shows the time taken to travel 120 km at different speeds.*

| *Speed, x (km/h)* | *20* | *30* | *40* | *60* | *80* | *120* |
|---|---|---|---|---|---|---|
| *Time, t (h)* | *6* | *4* | *3* | *2* | *1.5* | *1.0* |

*Table 7.9*

*(a) What happens to the time when speed is doubled?*

*(b) What happens to the time when speed is trebled?*

*(c) If the speed increases in the ratio 3 : 2, how does the time change?*

*(d) If the speed increases in the ratio 4 : 3, how does the time change?*

*(e) What do you notice about the product xt?*

## *Solution*

*(a) Time is halved.*

*(b) Time is* $\frac{1}{3}$ *the original time.*

*(c) Time decreases in the ratio 2 : 3.*

*(d) Time decreases in the ratio 3 : 4.*

*(e) xt is a constant i.e. xt = 120.*

From Example 7.7, we notice that

If quantities $r$ and $s$ are **inversely proportional** , $rs$ is a constant and when one of $r$ and $s$ increases in the ratio $\frac{a}{b}$ , the other decreases in the ratio $\frac{b}{a}$ .

## Exercise 7.4

1. Julie drives from Mzuzu to Mzimba for 5 h at 80 km/h. How long does she take if she drives at 60 km/h?
2. Four tractors can plough a farm in 8 days. How long will it take to plough the farm using three tractors?
3. Twelve women weed a field in 10 days. How long do 15 women take?
4. Given that x and y are inversely proportional, copy and complete Table 7.10 .

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 5 | 10 | 15 | 20 | – |
| $y$ | – | 38 | – | – | 9.5 |

*Table 7.10*

5. A race cyclist takes 50 min for a race if he cycles at 48 km/h. At what speed must he cycle to do the race in 40 min?
6. A camp has enough food for 300 refugees for 6 days. How long will the food last if 50 more refugees are brought in?
7. Three men working 10 hours a day can do a certain job in 8 days. How long would 2 men working 12 hours a day take to do the same job?
8. Nineteen men working $7\frac{1}{2}$ hours a day recarpeted a section of a road in 21 days. How many hours a day must 45 men work in order to recarpet a similar section of the road in 7 days?

## Graphical representation of inverse proportion

Suppose a motorist is to cover a fixed distance, driving non-stop. We know that the faster he drives, the shorter the time he takes.

Suppose the distance is 240 km.

If his speed is 40 km/h, he takes

$\frac{240}{40}$ = 6 hours.

If his speed is 60 km/h, he takes

$\frac{240}{60}$ = 4 hours.

If his speed is 80 km/h, he takes

$\frac{240}{80}$ = 3 hours.

If his speed is 120 km/h, he takes

$\frac{240}{120}$ = 2 hours, and so on.

In this case, the higher the speed, the shorter the time taken to cover the same distance. The speed is said to **vary inversely** (or **indirectly** ) as time.

If S represents speed and T represents time in hours, then ‘S varies inversely as T’ is written as

$S \propto \frac{1}{T}$ .

Just as in direct proportion, $S \propto \frac{1}{T}$ means that ST is a constant.

i.e. $S \times \frac{T}{1}$ = constant

or ST = constant.

If $S \propto \frac{1}{T}$ , then ST = $k$ where $k$ is a constant.

**Note:** If the values of S are plotted against the corresponding values of T, the resulting graph would be a curve as in Fig. 7.2 .

This graph shows that as the values of T increase, the values of S decrease (approaching zero).

Similarly, as the values of S increase (towards 240 km), the values of T decrease.

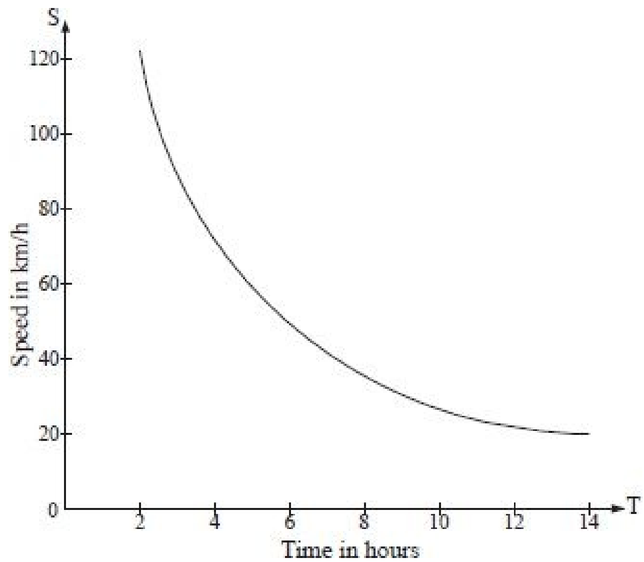



*Fig. 7.2*

Now construct a table of values giving corresponding values of S and $\frac{1}{T}$ for the motorist travelling a distance of 240 km. Draw the graph of S against $\frac{1}{T}$.
Describe fully the resulting graph.

In general, if *y* varies inversely as the *n* th power of *x*, written as

$$y \propto \frac{1}{x^n}$$

then $y \div \frac{1}{x^n}$ is a constant

i.e. $\frac{y}{\frac{1}{x^n}} = c$

$\Rightarrow \quad yx^n = c$

or $\quad y = \frac{c}{x^n}$

The graph of *y* against $\frac{1}{x^n}$ is a straight line.

## *Example 7.8*

*The number of beats per minute of a pendulum varies inversely as the square root of its length. If a pendulum which is 49 cm long makes 36 beats per minute, how many beats per minute would a pendulum which is 144 cm long make?*

### *Solution*

*Let the length of the pendulum be l cm and the number of beats per minute be n.*

*Thus,* $n \propto \frac{1}{\sqrt{l}}$ .

*i.e,* $n = c\frac{1}{\sqrt{l}}$ *, where c is a constant*

$\therefore \quad n\sqrt{l} = c$

*When* $l = 49$*, and* $n = 36$*,*

$36\sqrt{49} = c$

$\Rightarrow c = 252$ *(only the positive* $\sqrt{49}$ *is meaningful)*

$\therefore \quad n\sqrt{l} = 252$

*When* $l = 144$

$$\begin{aligned} n\sqrt{144} &= 252 \\ \therefore \quad n &= \frac{252}{\sqrt{144}} \\ &= \frac{252}{12} \\ &= 21 \end{aligned}$$

*Thus, a pendulum 144 cm long makes 21 beats per minute.*

## Exercise 7.5

1. Write down the relationship between the variables given below.
   (a) *a* varies directly as *b.*
   (b) *m* is proportional to *n.*
   (c) *x* is inversely proportional to *y.*
   (d) P varies inversely as *y.*
   (e) R is proportional to *m* and *p.*
   (f) *w* varies directly as the square of *x* and inversely as *y.*
   (g) *c* varies directly as *d* and inversely as *g* and the square of *f* .

2. If *y* varies inversely as *x* and $y = 12$ when $x = 6$, find *y* in terms of *x*. Use a graph to find *y* when $x = 8$ and *x* when $y = \frac{1}{2}$ .

3. *y* varies inversely as the square of *x. If* $y = 9$ when $x = 10$, find *y* in terms of *x.* Use a graph to find *y* when $x = 6$ and *x* when $y = 4$.

# 8 INDICES AND LOGARITHMS

## Indices

In Form 1, we learnt to express numbers as products of their prime factors.

For example, 8 can be expressed as,

8 = 2 x 2 x 2 (prime factors)

We can write the prime factors of 8 as 23, where 23 stands for $\underbrace{2 \times 2 \times 2}_{\text{3 factors}}$

Similarly,

$3^5$ stands for $\underbrace{3 \times 3 \times 3 \times 3 \times 3}_{\text{5 factors}}$, and so on.

Thus, $a^4$ stands for $\underbrace{a \times a \times a \times a}_{\text{4 factors}}$

The raised numeral is called an **index** (plural: **indices** ) or **power** or **exponent** .

For example, when we write $64 = 4 \times 4 \times 4 = 4^3$ , 3 is called the index (or power or exponent) and 4 is called the base. We say that 64 is the third power of 4.

$4^3$ is the index form (to base 4) of 64.

In general,

If $n = a^x$

$$= \underbrace{a \times a \times \ldots \times a}_{x \text{ factors}}$$

where $x$ is a positive integer, $a^x$ is the index form of $n$, where $a$ is the base and $x$ is the index or power.

### *Example 8.1*

*(a) Express $5 \times 5 \times 5 \times 5 \times 5 \times 5$ in index form.*

*(b) Write $4^5$ in expanded form.*

(c) Evaluate $2^4$ .

**Solution**

(a) In $\underbrace{5 \times 5 \times 5 \times 5 \times 5 \times 5}_{6 \text{ factors}}$ 5 is the base and 6 is the index

$\therefore 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$

(b) In $4^5$ , 4 is the base and 5 is the index

$\therefore \quad 4^5 = \underbrace{4 \times 4 \times 4 \times 4 \times 4}_{5 \text{ factors}}$

(c) In $2^4$ , the index 4 indicates that four 2s are multiplied together.

$\therefore 2^4 = 2 \times 2 \times 2 \times 2$

$= 16$

## Exercise 8.1

1. Express each of the following in index form.
   (a) $3 \times 3$
   (b) $2 \times 2$
   (c) $4 \times 4 \times 4 \times 4$
   (d) $2 \times 2 \times 2 \times 2 \times 2$
   (e) $6 \times 6 \times 6 \times 6$
   (f) $9 \times 9 \times 9 \times 9 \times 9 \times 9$
   (g) $11 \times 11 \times 11 \times 11 \times 11$
   (h) $\underbrace{13 \times 13 \times \ldots \times 13}_{12 \text{ factors}}$
2. Write the following in expanded form.
   (a) $2^3$
   (b) $3^4$
   (c) $3^7$
   (d) $4^3$
   (e) $7^3$

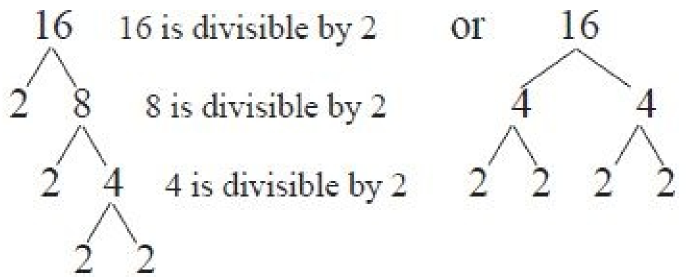
(f) $5^6$
(g) $15^6$
(h) $12^3$

3. Copy and complete:
   $4^2 = 16$. Thus 16 is the second power of 4.
   $3^3 =$ __. Thus __ is the third power of 3.
   $6^3 =$ __. Thus __ is the __ power of __.
4. Which is greater, $5^3$ or $3^5$ ?
5. How many factors are there in
   (a) 2
   (b) $2^4$
   (c) $2^{10}$ ?
6. Write in short form using index notation.
   (a) $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$
   (b) $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$
   (c) $b \times b \times b \times b \times b \times b$
   (d) $m \times m \times m \times m$
   (e) $\underbrace{t \times t \times \ldots \times t}_{\text{20 factors}}$

## Writing numbers in index form

We have seen that we can write 4 as $2^2$ , 25 as $5^2$ , and so on. Thus it should be possible to write any given number in index form in the simplest form as long as the number has factors.

For example, 16 can be factorised using a factor tree as shown below.




$$\therefore\ 16 = 2 \times 8 \qquad \text{Also } 16 = \underbrace{4 \times 4}_{\text{same factors}}$$

$$= 2 \times 2 \times 4 \qquad = 4^2$$

$$= \underbrace{2 \times 2 \times 2 \times 2}_{\text{same factors}}$$

$$= 2^4 \text{ (simplest form)}$$

Note that $2^4$ is considered to be a simpler form since in $4^2$ , 4 is not a prime number.

## *Example 8.2*

*Write each of the following in its simplest index form.*

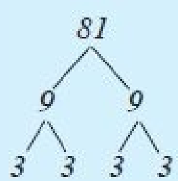
*(a) 81*

*(b) 96*

*(c) $5 \times c \times c \times 5 \times c \times 5$*

## *Solution*

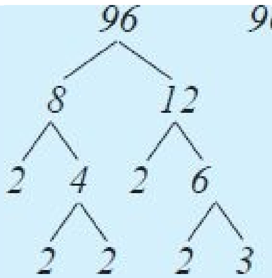
*(a)*



*$81 = 9^2$ (This is not the simplest form since 9 is not a prime number)*

$$\therefore\ 81 = \underbrace{3 \times 3 \times 3 \times 3}_{4\ factors}$$

$$= 3^4 \text{ (This is the simplest index form)}$$

(b) 


$$96 = \underbrace{2 \times 2 \times 2 \times 2 \times 2}_{\text{same factors}} \times \underbrace{3}_{\text{single factor}}$$

$$= 2^5 \times 3$$

(c) $5 \times c \times c \times 5 \times c \times 5$

$$= \underbrace{5 \times 5 \times 5}_{\text{same factors}} \times \underbrace{c \times c \times c}_{\text{same factors}} \quad \text{(Rearranging the factor)}$$

$$= 5^3 \times c^3$$

$$= 5^3c^3$$

## Exercise 8.2

1. Write each of the following in index form using the specified base.
   - (a) 25 (base 5)
   - (b) 64 (base 4)
   - (c) 49 (base 7)
   - (d) 1 000 (base 10)
2. Write each of the following in its simplest index form.
   - (a) $2 \times a \times a \times a$
   - (b) $3 \times y \times y$
   - (c) $h \times h \times h \times 7 \times h \times 21$
   - (d) $3 \times b \times b \times a \times b \times b \times b$
   - (e) $3 \times a \times 3 \times a \times a \times a$
3. Write each of the following in its simplest index form.
   - (a) 2
   - (b) 8
   - (c) 32
   - (d) 16
   - (e) 64
   - (f) 128
4. Evaluate
   - (a) $2^2 \times 3^3$

(b) $2^4 \times 3^2$
(c) $4^2 \times 3^2$
(d) $5^2 \times 2^2$
(e) $2^3 \times 10^4$
(f) $4^3 \times 10^5$

5. Evaluate the following if $a = 2$.
   (a) $3a^2$
   (b) $5a^3$
   (c) $9a^2$
   (d) $6a^2$
   (e) $16a^2$

6. If $n$ is a whole number, find the smallest value of $n$ for which $a^n$ is greater than 35 given that
   (a) $a = 2$
   (b) $a = 3$
   (c) $a = 4$

7. Express $2 \times 2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 5 \times 5$ in index form.

8. Express 72 and 108 as products of powers of 2 and 3.

## Laws of indices

When numbers are in index form, they are easier to write and also to use in calculations.
The following laws of indices are used in calculations involving multiplication, division, power and roots.

## Multiplication of numbers in index form

We have already seen that

$$2^3 = \underbrace{2 \times 2 \times 2}_{3 \text{ factors}} \quad \text{and} \quad 2^5 = \underbrace{2 \times 2 \times 2 \times 2 \times 2}_{5 \text{ factors}}$$

$$\begin{aligned}
\text{Thus, } 2^3 \times 2^5 &= \underbrace{(2 \times 2 \times 2)}_{3 \text{ factors}} \times \underbrace{(2 \times 2 \times 2 \times 2 \times 2)}_{5 \text{ factors}} \\
&= \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}_{(3+5) \text{ same factors}} \\
&= 2^8 \\
\therefore\ 2^3 \times 2^5 &= 2^{(3+5)} \\
&= 2^8
\end{aligned}$$

***Example 8.3***

*Simplify each of the following giving your answer in index form.*

(a) $10^2 \times 10^5$

(b) $2^3 \times 2^7$

***Solution***

(a) $10^2 \times 10^5$

$$\begin{aligned}
&= \underbrace{(10 \times 10)}_{2 \text{ factors}} \times \underbrace{(10 \times 10 \times 10 \times 10 \times 10)}_{5 \text{ factors}} \\
&= \underbrace{10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10}_{(2+5) \text{ factors}} \\
&= 10^{(2+5)} \\
&= 10^7
\end{aligned}$$

(d) $2^3 \times 2^7$

$= (2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2)$

$= 2(3 + 7)$

$= 2^{10}$

When numbers, written in index form with a common base, are multiplied, the indices are added while the base remains unaltered,

i.e. $a^x \times a^y = a^{(x+y)}$

If in a multiplication there is more than one letter to be multiplied, they must be multiplied separately because each represents a different value.

### *Example 8.4*

*Simplify $4x^3 y^3 \times 5x^4 y^5$*

### *Solution*

$4x^3 y^3 \times 5x^4 y^5 = 4 \times 5 \times x^3 \times x^4 \times y^3 \times y^5$

*(Rearranging the factors)*

$= 20 \times x^{(3+4)} \times y^{(3+5)}$

$= 20 \times x^7 \times y^8$

$= 20x^7 y^8$

Squaring an expression simply means multiplying the expression by itself. For example,

$(a^5)^2 = a^5 \times a^5$

$= a^{(5+5)}$ (multiplication law)

$= a^{(5 \times 2)}$

$= a^{10}$

Thus, in order to square an algebraic expression, square the coefficient and double the indices of the letters.

### *Example 8.5*

*Find the square of $4x^4 y^3$.*

### *Solution*

$(4x^4 y^3)^2 = 4^2 \times x^{4 \times 2} \times y^{3 \times 2}$

$= 16x^8 y^6$

Or $(4x^4 y^3)^2 = 4x^4 y^3 \times 4x^4 y^3$

$= 4 \times 4 \times x^4 \times x^4 \times y^3 \times y^3$

$= 16x^8 y^6$ (adding indices of terms with common bases)

## Exercise 8.3

1. Simplify

   (a) $a^2 \times a^4$
   (b) $n^5 \times n^7$
   (c) $p^3 \times p^5$
   (d) $5^8 \times 5^5$
   (e) $p^3 \times p^4 \times p^5$
   (f) $z^7 \times z^{12} \times z$
   (g) $t^3 \times t^7 \times t^2$

2. Evaluate, leaving your answers in index form:

   (a) $2 \times 2^3$
   (b) $3^2 \times 3^3$
   (c) $3^2 \times 7^2 \times 3$
   (d) $10^3 \times 10^5$

3. Simplify

   (a) $4x^7 y^2 \times 2xy^3 z^2$
   (b) $2x^4 \times 5x^3$
   (c) $6x^2 y \times 3x^3 y^5$
   (d) $2a^2 b \times 4a^3 b^2$

4. Simplify

   (a) $(x^3)^2$
   (b) $(a^2 b^3)^2$
   (c) $(5a^3 bc^2)^2$
   (d) $(9a^5 b^5 c^2)^2$

## Division of numbers in index form

In factor form, $2^5 = 2 \times 2 \times 2 \times 2 \times 2$

and $2^3 = 2 \times 2 \times 2$

$$\text{Thus,} \quad 2^5 \div 2^3 = \frac{2^5}{2^3}$$

$$= \frac{\overbrace{2 \times 2 \times \cancel{2} \times \cancel{2} \times \cancel{2}}^{5 \text{ factors}}}{\underbrace{\cancel{2} \times \cancel{2} \times \cancel{2}}_{3 \text{ factors}}}$$

$$= \underbrace{2 \times 2}_{(5-3) \text{ factors}}$$

$$= 2^2$$

$$\therefore 2^5 \div 2^3 = 2^{(5-3)}$$

$$= 2^2$$

In this example, the power of 2 in the answer is the difference between that of the numerator and that of the denominator.

## *Example 8.6*

*Simplify each of the following giving your answer in the simplest index form.*

(a) $3^7 \div 3^4$

(b) $x^8 \div x^3$

## *Solution*

$$(a)\ 3^7 \div 3^4 = \frac{3^7}{3^4} = \frac{\overbrace{3 \times 3 \times 3 \times \cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3}}^{7\ factors}}{\underbrace{\cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3}}_{4\ factors}}$$

$$= \underbrace{3 \times 3 \times 3}_{(7-4)\ factors}$$

$$= 3^{(7-4)}$$

$$= 3^3$$

$$(a)\ x^8 \div x^3 = \frac{x^8}{x^3} = \frac{\overbrace{x \times x \times x \times x \times x \times \cancel{x} \times \cancel{x} \times \cancel{x}}^{8\ factors}}{\underbrace{\cancel{x} \times \cancel{x} \times \cancel{x}}_{3\ factors}}$$

$$= \underbrace{x \times x \times x \times x \times x}_{(8-3)\ factors}$$

$$= x^{(8-3)}$$

$$= x^5$$

If there is more than one letter to be divided, they must be divided separately as they represent different values.

### *Example 8.7*

*Simplify* $12x^4\ y^5 \div 3x^3\ y^2$

### *Solution*

$$12x^4y^5 \div 3x^3y^2 = \frac{12x^4y^5}{3x^3y^2}$$

$$= \frac{4x^4y^5}{x^3y^2} \quad \textit{(Dividing the coefficients)}$$

$$= \frac{4xy^5}{y^2} \quad \textit{(Dividing by } x^3\textit{)}$$

$$= 4xy^3 \quad \textit{(Dividing by } y^2\textit{)}$$

$$\therefore\ 12x^4y^5 \div 3x^3y^2 = 4xy^3$$

When two numbers, written in index form with a common base are divided, the indices are subtracted, i.e. $a^x \div a^y = a^{(x-y)}$

## Exercise 8.4

1. Use the division law of indices to evaluate
   (a) $2^6 \div 2^3$
   (b) $2^8 \div 2^5$
   (c) $2^{18} \div 2^{15}$
   (d) $4^{19} \div 4^{18}$
   (e) $2^{13} \div 2^8$
   (f) $2^9 \div 2^4$
   (g) $2^{10} \div 2^8$
   (h) $5^3 \div 5$
2. Simplify
   (a) $\frac{4^6}{4^4}$
   (b) $\frac{4^7}{4^3}$

(c) $\frac{10^9}{10^7}$

(d) $\frac{h^{11}}{h}$

(e) $\frac{a^8}{a^3}$

(f) $\frac{g^{12}}{g^9}$

(g) $\frac{g^6}{g}$

(h) $\frac{p^5}{p}$

(i) $\frac{h^9}{h^7}$

(j) $\frac{a^6}{a^4}$

3. Simplify

(a) $8a^4 \div 4a^{2^{\circ}}$

(b) $\frac{x^{12}y^5}{x^7y^4}$

(c) $4x^2y^5 \div 2xy^3$

(d) $\frac{35a^7b^{12}c^3}{5a^5b^4c^2}$

(e) $14p^9q^6r^2 \div 2pq$

4. If $A = 27x^4y^3z^4$ and $B = 3x^2yz^2$, find

(a) AB

(b) A ÷ B

(c) $A^2$

(d) $B^2$

## The zero index

Consider $3^4 \div 3^4$.

Using the division law of indices,

$$3^4 \div 3^4 = 3^{(4-4)} = 3^0$$

Using factor form,

$$3^4 \div 3^4 = \frac{3^4}{3^4} = \frac{\overset{1}{\cancel{3}} \times \overset{1}{\cancel{3}} \times \overset{1}{\cancel{3}} \times \overset{1}{\cancel{3}}}{\underset{1}{\cancel{3}} \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{3}} \times \underset{1}{\cancel{3}}} = 1$$

$$\therefore 3^0 = 1$$

Similarly,

$$x^3 \div x^3 = x^{(3-3)} = x^0$$

and $\dfrac{\cancel{x} \times \cancel{x} \times \cancel{x}}{\cancel{x} \times \cancel{x} \times \cancel{x}} = 1$

$$\therefore x^0 = 1$$

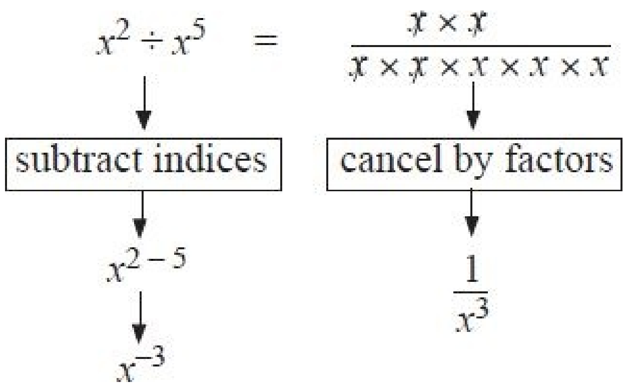
Any non-zero number raised to power zero, equals 1 i.e. $\boldsymbol{a^0 = 1}$ for all values of $a$.

## Negative indices

To divide $x^5$ by $x^2$, we simply subtract the indices

i.e. $x^5 \div x^2 = x^{5-2} = x^3$.

Now consider $x^2 \div x^5$.




Thus, $x^{-3}$ $\frac{1}{x^3}$ (Since we are doing the same thing but using different methods, the results must be equal)

Similarly, $x^{-2} = \frac{1}{x^2}$, $x^{-5} = \frac{1}{x^5}$, and so on.

Any number raised to a negative power is the same as the reciprocal of the equivalent positive power of the same number,

i.e. $a^{-x} = \frac{1}{a^x}$ and not $^{-}a^x$ , provided $a \neq 0$.

***Example 8.8***

*Simplify the following giving answers with positive indices.*

(a) $x^5 \div x^8$

(b) $x^3 \div x^{-4}$

***Solution***

(a) $x^5 \div x^8 = x^{5-8} = x^{-3}$

$= \frac{1}{x^3}$

(b) $x^3 \div x^{-4} = x^{3-(-4)} = x^{3+4}$

$= x^7$

*Or*

$$x^3 \div x^{-4} = \frac{x^3}{x^{-4}} = \frac{x^3}{\frac{1}{x^4}}$$

$$= x^3 \times x^4$$

$$= x^{3+4} = x^7$$

## Other integral powers (powers of powers)

Consider $(2^2)^3$ .

This means that '2 squared is cubed'.

Thus, $(2^2)^3 = 2^2 \times 2^2 \times 2^2$

$= 2^{2+2+2}$

$= 2^6$

But $2 + 2 + 2 = 2 \times 3$

So, in evaluating $(2^2)^3$, we have in fact multiplied the indices together.

Similarly, $(x^5)^4 = x^5 \times x^5 \times x^5 \times x^5$

$= x^{5+5+5+5}$

$= x^{5 \times 4}$

$= x^{20}$

When a number, written in index form, is raised to another power, the indices are multiplied,

i.e. $(a^x)^y = a^{x \times y} = a^{xy}$.

Now consider $(2 \times 3)^3$.

$(2 \times 3)^3$ means '(2 × 3) cubed'.

Thus, $(2 \times 3)^3 = (2 \times 3) \times (2 \times 3) \times (2 \times 3)$

$= 2 \times 2 \times 2 \times 3 \times 3 \times 3$

(regrouping like factors)

$= 2^3 \times 3^3$.

Similarly, $(a \times b)^3 = a^3 b^3$.

All the numerals which are multiplied together in a bracket are raised to the power of that bracket

i.e. $(a \times b)^x = a^x bx$.

$(\frac{2}{3})^3$ means $\frac{2}{3}$ is 'cubed'

$$(\tfrac{2}{3})^3 = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$$

$$= \frac{2 \times 2 \times 2}{3 \times 3 \times 3}$$

$$= \frac{2^3}{3^3}$$

Similarly, $(\frac{a}{b})^4 = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}$

$$= \frac{a \times a \times a \times a}{b \times b \times b \times b}$$

$$= \frac{a^4}{b^4}$$

All the numerals which are divided in a bracket are raised to the power of the bracket

i.e. $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$.

## Exercise 8.5

1. Simplify
   - (a) $2^0$
   - (b) $x^1$
   - (c) $100^0$
   - (d) $(4x)^1$
   - (e) $(xy)^0$
   - (f) $3x^0$
   - (g) $5^2 x^0 y^0$
   - (h) $(3a)^0 + 3$
   - (i) $4a^0 - 4$
2. Write the following with positive indices and then simplify.
   - (a) $a^{-4} \times a$
   - (b) $m^{12} \times m^{-2} \div m^{-5}$
   - (c) $2x^{-5} \times x$

(d) $x \times \frac{1}{x^{-2}}$

3. Work out the following.

(a) $(x^4)^2 \times x^2$
(b) $(a^5)^3 \times a^{-5}$
(c) $(y^7)^2 \div y^{14}$
(d) $(a^3)^4 \times a^{-12}$
(e) $(x^{-2})^6 \times x^{13}$
(f) $a^8 \times (a^3)^2$

4. Simplify

(a) $(3 \times 5)^2$
(b) $(2ab)^3$
(c) $(72xy)^0$
(d) $(x^2y^2)^3$

5. Simplify, giving your answers in positive index form.

(a) $\left(\frac{3}{5}\right)^2$
(b) $\left(\frac{x}{y}\right)^5$
(c) $\left(\frac{m}{n}\right)^{-2}$
(d) $\left(\frac{p}{q}\right)^3$
(e) $\left(\frac{a}{b}\right)^{-3}$
(f) $\left(\frac{l}{m}\right)^{-1}$
(g) $\left(\frac{3}{7}\right)^3$
(h) $\left(\frac{1}{8}\right)^{-2}$
(i) $\left(\frac{x}{y}\right)^7$

6. Simplify

(a) $\left(\frac{x^2}{y}\right)^3$
(b) $\left(\frac{n^2}{m}\right)^{-2}$
(c) $(x^3)^3 \div x^5$
(d) $\left(\frac{ab^2}{c}\right)^5$

7. Evaluate

(a) $6^0$

(b) $1\ 000^0$
(c) $10^9 \times 10^{-8}$
(d) $7^3 \div 7^4$
(e) $\left(\frac{3}{4}\right)^{-2}$
(f) $(2 \times 5)^{-1}$
(g) $(3^3)^2$
(h) $(4^2)^0$

8. Write the following with negative indices.
(a) $\frac{1}{6^2}$
(b) $\frac{1}{y^4}$
(c) $\frac{1}{2^9}$
(d) $\frac{1}{z^8}$
(e) $\frac{1}{a^3}$
(f) $\frac{1}{2x^5}$

9. Write the following using positive indices.
(a) $x^{-6}$
(b) $y^{-3}$
(c) $2^{-1}$
(d) $p^{-8}$
(e) $q^{-4}$
(f) $3^4 x^{-2}$

10. Find the value of each of the following:
(a) $x^{-1}$ if $x = 11$
(b) $z^0$ if $z = 3$
(c) $x^{-5}$ if $x = 2$
(d) $p^{-2}$ if $p = 9$
(e) $x^{-2}$ if $x = 12$

11. If $x$ is greater than 1, which is greater, $x$ or $x^2$ ?

12. If $0 < x < 1$, which is greater, $x$ or $x^2$ ?

13. If you were told that $a^8 < a^4$ , what would you say about $a$ ?

14. If $0 < x < 1$, arrange $x^2$, $x^4$, $x^3$ in ascending order.

## Fractional indices

We learnt that if $n$ is any positive integer, $a^n$ is the short form for

$$\underbrace{a \times a \times \ldots \times a}_{n \text{ factors}}$$

We know that fractional powers obey the same laws of indices as integral powers. What then is the meaning of

(a) $4^{\frac{1}{2}}$

(b) $8^{\frac{1}{3}}$

(c) $8^{\frac{2}{3}}$ ?

Consider

$$4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^{\frac{1}{2}+\frac{1}{2}} \quad \text{(Addition law of indices)}$$
$$= 4^1 = 4$$

$\therefore$ $4^{\frac{1}{2}}$ is the number which when multiplied by itself gives 4.

Also we know that $\sqrt{4} \times \sqrt{4} = 2 \times 2 = 4$

$$\therefore \quad 4^{\frac{1}{2}} = \sqrt{4}$$

Similarly, $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$

and $\sqrt{a} \times \sqrt{a} = a$

$\therefore$ $a^{\frac{1}{2}} = \sqrt{a}$ for all positive values of $a$

Consider

$$8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 8^1 = 8$$

$\therefore$ $8^{\frac{1}{3}}$ is the number which when multiplied by itself three times gives 8.

Also $\sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} = 2 \times 2 \times 2 = 8$

$$\therefore \quad 8^{\frac{1}{3}} = \sqrt[3]{8}$$

Similarly, $\left(a^{\frac{1}{3}}\right)^3 = a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}}$

$= a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a^1$

or $\left(a^{\frac{1}{3}}\right)^3 = a^1 \left(\text{since } \left(a^{\frac{1}{3}}\right)^3 = a^{\frac{1}{3}\times 3}\right)$

and $\left(\sqrt[3]{a}\right)^3 = a$ (cube root of a number cubed gives the number itself).

$\left(a^{\frac{1}{3}}\right) = \sqrt[3]{a}$ for all values of *a*.

We have seen that $a^{\frac{1}{2}} = \sqrt{a}$ and $a^{\frac{1}{3}} = \sqrt[3]{a}$ . So in general, $a^{\frac{1}{n}} = \sqrt[n]{a}$, , where *n* is a positive integer.

**Note:** $\sqrt[3]{a}$ means cube root or 3rd root of *a*. Likewise, $\sqrt[n]{a}$ means the *n* th root of *a*
*n* is called the order of the root.
Consider

$$8^{\frac{2}{3}} = 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \text{ since } \tfrac{2}{3} = \tfrac{1}{3} + \tfrac{1}{3}$$
$$= \left(8^{\frac{1}{3}}\right)^2 \text{ since } \left(8^{\frac{1}{3}}\right)^2 = 8^{\frac{1}{3}\times 2} = 8^{\frac{2}{3}}$$
$$= \left(\sqrt[3]{8}\right)^2 \text{ since } 8^{\frac{1}{3}} = \sqrt[3]{8}$$
$$= 2^2$$
$$= 4$$

Similarly, $a^{\frac{2}{3}} = \left(a^{\frac{1}{3}}\right)^2$

$= \left(\sqrt[3]{a}\right)^2$

Since $\sqrt[3]{a}$ can also be written as $(a^2)^{\frac{1}{3}}$ , then

$a^{\frac{2}{3}} = \left(\sqrt[3]{a}\right)^2 = \sqrt[3]{a^2}$

## *Example 8.9*

*Find the square root of $16x^4 y^2$*

*(a) by factor method, and*

*(b) using laws of indices.*

***Solution***

(a) $16x^4 y^2 = 2 \times 2 \times 2 \times 2 \times x \times x \times x \times x \times y \times y$

*(Expanded form)*

$= \overline{2 \times 2} \times \overline{2 \times 2} \times \overline{x \times x} \times \overline{x \times x} \times \overline{y \times y}$

*(Pairing off factors)*

$\therefore \sqrt{16x^4y^2} = 2 \times 2 \times x \times x \times y$

*(Picking one out of every two like factors)*

$= 2^2 \times x^2 \times y$

$= 4x^2 y$

(b)

$\sqrt{16x^4y^2} = (16x^4y^2)^{\frac{1}{2}}$ *(since $a^{\frac{1}{2}} = \sqrt{a}$)*

$= 16^{\frac{1}{2}}(x^4)^{\frac{1}{2}}(y^2)^{\frac{1}{2}}$ *(Raising each of the factors to power $\frac{1}{2}$ )*

$= 4x^2y$ *(since $(a^m)^n = a^{mn}$)*

Note that method (b) is quicker.

From Example 8.9, we see that to find the square root of an algebraic expression, we find the square root of the coefficient and divide the indices of the letters by 2,

e.g. $\sqrt{81a^4b^{12}} = 9a^2 b^6$

Similarly, for any root of order $n$, we simply find the $n$ th root of the coefficient and divide the indices of the letters by the order $n$ of the root,

e.g. $\sqrt[4]{81a^4b^{12}}$

***Example 8.10***

*Evaluate* $16^{\frac{3}{4}}$

***Solution***

$$\begin{aligned} 16^{\frac{3}{4}} &= \left(16^{\frac{1}{4}}\right)^3 \text{ since } \tfrac{1}{4} \times 3 = \tfrac{3}{4} \\ &= 16^{\frac{1}{4}} \times 16^{\frac{1}{4}} \times 16^{\frac{1}{4}} \\ &= \sqrt[4]{16} \times \sqrt[4]{16} \times \sqrt[4]{16} \text{ since } 16^{\frac{1}{4}} = \sqrt[4]{16} \\ &= 2 \times 2 \times 2 = 8 \end{aligned}$$

$$\begin{aligned} \text{Also } 16^{\frac{3}{4}} &= \left(16^3\right)^{\frac{1}{4}} \text{ since } 3 \times \tfrac{1}{4} = \tfrac{3}{4} \\ &= (16 \times 16 \times 16)^{\frac{1}{4}} \\ &= \left(2^4 \times 2^4 \times 2^4\right)^{\frac{1}{4}} = 2^{4 \times \frac{1}{4}} \times 2^{4 \times \frac{1}{4}} \times 2^{4 \times \frac{1}{4}} \\ &= 2 \times 2 \times 2 = 8 \end{aligned}$$

*In this example, it is easier to take the root first since it is exact.*

In general,

$$x^{\frac{a}{b}} = \left(x^a\right)^{\frac{1}{b}} = \sqrt[b]{x^a}$$

or

$$x^{\frac{a}{b}} = \left(x^{\frac{1}{b}}\right)^a = \left(\sqrt[b]{x}\right)^a \text{ (best when } \sqrt[b]{x} \text{ is exact).}$$

## Exercise 8.6

Evaluate the expressions in Questions 1-8.

1. (a) $8^{\frac{1}{3}}$

   (b) $9^{\frac{1}{2}}$

   (c) $\left(\frac{16}{25}\right)^{\frac{1}{2}}$

   (d) $\left(\frac{1}{81}\right)^{\frac{1}{4}}$

2. (a) $27^{\frac{2}{3}}$

   (b) $9^{\frac{3}{2}}$

(c) $32^{\frac{2}{5}}$

(d) $\left(81^{\frac{1}{2}}\right)^{\frac{3}{2}}$

3. (a) $16^{-\frac{1}{2}}$

   (b) $8^{-\frac{4}{3}}$

   (c) $\left(\frac{9}{16}\right)^{-\frac{1}{2}}$

   (d) $\left(6\frac{1}{4}\right)^{-\frac{1}{2}}$

   (e) $\left(\frac{49}{16}\right)^{-\frac{1}{2}}$

4. $8^{\frac{2}{3}} + 8^0 - 2^{-2}$

5. $81^{\frac{3}{4}} - \left(\frac{1}{5}\right)^{-1} - 27^0$

6. $\left(\frac{1}{27}\right)^{-\frac{1}{3}} + \left(\frac{1}{3}\right)^{-2} + 16^{\frac{1}{2}}$

7. $4^{\frac{1}{2}} \times 3^{-2} \div 8^{-\frac{2}{3}}$

8. $(x - 1)^{\frac{4}{3}} + (x + 7)^{\frac{3}{4}} + 3x^0$ when $x = 9$.

9. Express as a single power of 2.

   $8^{\frac{5}{3}} \times 64^{\frac{1}{2}} \times 4^{-3}$

10. Simplify

    (a) $\sqrt{9x^2y^4}$

    (b) $\sqrt{36n^4m^6}$

    (c) $\sqrt{3^4x^8y^6z^2}$

    (d) $\sqrt{64s^4t^{10}}$

## Standard form

Consider the number 6 520.

6 520 can be written down as 6.52 × 1 000

i.e. 6 520 = 6.52 × 1 000

$= 6.52 \times 10^3$

$6.52 \times 10^3$ is said to be the **standard form** of 6 520. Similarly, the number 0.000

15 can be written as

$$0.000\ 15 = \frac{1.5}{10\ 000}$$
$$= \frac{1.5}{10^4}$$
$$= 1.5 \times \frac{1}{10^4}$$
$$= 1.5 \times 10^{-4}$$

$1.5 \times 10^{-4}$ is said to be the **standard form** of 0.000 15.

A number is in standard form if it is expressed in the form $\boldsymbol{a} \times \mathbf{10}^{\boldsymbol{n}}$, where $a$ is a number between 1 and 10, and $n$ is an integer.

**Note:** Standard form is also known as **scientific notation** since in science large and small values are often written in this form.

### *Example 8.11*

*(a) Express 78 956 340 in standard form, correct to 4 s.f.*

*(b) Calculate $7.15 \times 10^2 \times 4 \times 10^3$ giving your answer in standard form.*

*(c) Evaluate $(1.2 \times 10^9) \div (4.8 \times 10^{-4})$.*

### *Solution*

*(a)* $78\ 956\ 340 = 7.895\ 634 \times 10\ 000\ 000$
$= 7.895\ 634 \times 10^7$
$= 7.896 \times 10^7$ *(4 s.f)*

*(b)* $7.15 \times 10^2 \times 4 \times 10^3$

*(Regrouping)*

$= 7.15 \times 4 \times 10^2 \times 10^3$

$= 28.60 \times 10^5$

$= 2.86 \times 10 \times 10^5$

*(Adding powers of 10)*

$= 2.86 \times 10^6$

(c) $(1.2 \times 10^9) \div (4.8 \times 10^{-4})$

$= \dfrac{1.2 \times 10^9}{4.8 \times 10^{-4}}$

$= \dfrac{12 \times 10^9}{48 \times 10^{-4}}$ *(Multiplying numerator and denominator each by 10)*

$= \frac{1}{4} \times 10^{9 - -4}$ *(Subtracting powers of 10)*

$= \frac{1}{4} \times 10^{13}$

$= 0.25 \times 10^{13}$

$= \frac{2.5}{10^1} \times 10^{13}$

$= 2.5 \times 10^{-1} \times 10^{13}$

$= 2.5 \times 10^{12}$ *(Adding powers of 10)*

## Exercise 8.7

1. Write the following in the ordinary form.

   (a) $1.2 \times 10^1$
   (b) $4.275 \times 10^0$
   (c) $8.863 \times 10^3$
   (d) $6.2 \times 10^3$
   (e) $9.578 \times 10^{-2}$
   (f) $3.94 \times 10^{-3}$
   (g) $7.859 \times 10^{-1}$
   (h) $3.65 \times 10^2$

2. Express the following numbers in standard form.

(a) 74.3
(b) 65.2
(c) 86.25
(d) 97.38
(e) 112.2
(f) 432.6
(g) 413
(h) 982
(i) 724.9
(j) 324.8
(k) 5 401
(l) 3 096
(m) 23 847
(n) 92 652
(o) 827 400

3. Write the following in the ordinary form.

(a) $2 \times 10^{-1}$
(b) $0.3 \times 10^{2}$
(c) $0.02 \times 10^{-2}$
(d) $0.158 \times 10^{-3}$

4. Express the following numbers in standard form.

(a) 0.2
(b) 0.03
(c) 0.001 5
(d) 0.105 02
(e) 0.000 526 8
(f) 0.000 045 8
(g) 0.003 84
(h) 0.001 056
(i) 0.468 9
(j) 0.003 97
(k) 0.001 25
(l) 0.000 000 914 8

## Logarithms

If a number is expressed in index form with 10 as the base, the index (or power) is called the **common logarithm** of the number or simply the **logarithm of the number** .

Thus, 100 = 102

⇒ 2 is the logarithm of 100 to base 10. This is written in short as

$$\underset{\text{base}}{\log_{10}}\ \ \underset{\text{number}}{100} = \underset{\text{logarithm}}{2}$$

**Note:** Common logarithms are usually written without indicating the base, e.g. log 100 means the same as $\log_{10}$ 100.

$100\ 000\ 000 = 10^8$

$\therefore \log 10^8 = 8$

Likewise,

$0.000\ 001 = \frac{1}{10^6} = 10^{-6}$

$\therefore \log 10^{-6} = {}^{-}6$

Table 8.1 below shows logarithms of some powers of 10.

| Number | Power of 10 | Logarithm |
|---|---|---|
| $\frac{1}{1\ 000\ 000}$ | $\frac{1}{10^6} = 10^{-6}$ | $^-6$ |
| $\frac{1}{100\ 000}$ | $\frac{1}{10^5} = 10^{-5}$ | $^-5$ |
| $\frac{1}{10\ 000}$ | $\frac{1}{10^4} = 10^{-4}$ | $^-4$ |
| $\frac{1}{1\ 000}$ | $\frac{1}{10^3} = 10^{-3}$ | $^-3$ |
| $\frac{1}{100}$ | $\frac{1}{10^2} = 10^{-2}$ | $^-2$ |
| $\frac{1}{10}$ | $\frac{1}{10^1} = 10^{-1}$ | $^-1$ |
| 1 | $1 = 10^0$ | 0 |
| 10 | $1 = 10^1$ | 1 |
| 100 | $1 = 10^2$ | 2 |
| 1 000 | $1 = 10^3$ | 3 |
| 10 000 | $1 = 10^4$ | 4 |
| 100 000 | $1 = 10^5$ | 5 |
| 1 000 000 | $1 = 10^6$ | 6 |

*Table 8.1*

In general, the logarithm of any power of 10 is equal to the power or index, i.e. $\log 10^{n} = n$ and $\log 10^{-n} = {}^{-}n$ .

## Logarithms of numbers between 1 and 10

We know that $1 = 10^0$ . Thus, $\log 1 = 0$.

Also, $10 = 10^1$ . Hence, $\log 10 = 1$.

From this we observe that if a number lies between 1 and 10, its logarithm must lie between 0 and 1. Such a number is not an exact power of 10. Its logarithm, therefore, can only be approximated.

**Tables of logarithms** (found in books of Mathematical Tables), give approximate values of logarithms of numbers between 1 and 10. Many tables give these values correct to 4 significant figures, hence the name '4-figure table'.

The following examples illustrate how to find logarithms of numbers using logarithm tables.

## *Example 8.12*

*Find the logarithm of 3.25 using the table of logarithms.*

## *Solution*

*Look for 3.2 in the first column and move across the page along the row of 3.2 until you get to the column headed 5 in the main columns (Table 8.2 ).*

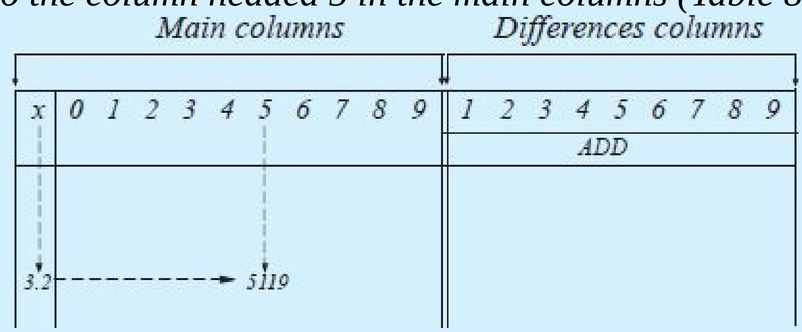



*Table 8.2*

*The number at the intersection of row 3.2 and column 5 is 5 119, and will lead to the logarithm of 3.25.*

*Since 3.25 lies between 1 and 10, its logarithm must lie between 0 and 1.*

*∴ $Log_{10}$ 3.25 = 0.511 9 (the d.p. is located by inspection according to the size of the number).*

*⇒ $10^{0.511\ 9} = 3.25$*

## *Example 8.13*

*Use tables to find the logarithm of 3.253 8.*

## *Solution*

*3.253 8 ≈ 3.254 (4 s.f. since tables cater for only 4 s.f.)*

*Look for 3.2 in the first column and move across the page along the row until* *you get to the column headed 5 in the main columns (Table 8.3 ).*

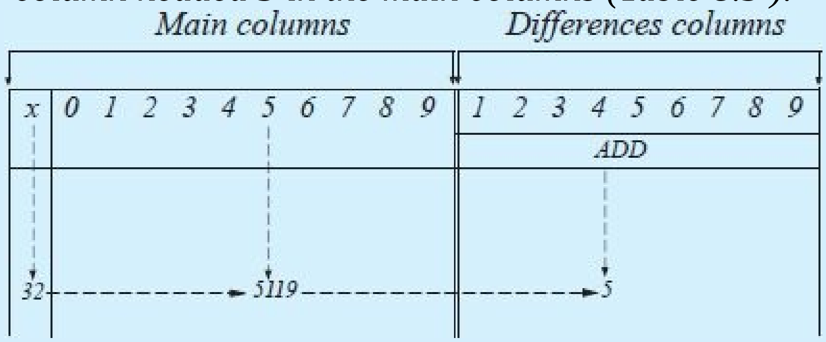



*Table 8.3*

*The number at the intersection of the row and the column is 5 119.*

*Continue across the page along the same row until you get to the differences column headed 4. The number at this intersection is 5.*

*The number 5 119 will lead to the logarithm of 3.25. But we require the logarithm of 3.254.*

*∴ Add the difference 5 to 5 119 to get 5 124*

*∴ Log 3.253 8 = 0.512 4 (since 3.254 lies between 1 and 10)*

### *Example 8.14*

*Find the logarithm of 5.034 356.*

### *Solution*

*5.034 356 ≈ 5.034 (4 s.f. since tables cater for only 4 s.f.)*

*Row 5.0 and main columns 3 give 7 016. Differences column 4 gives 4.*

*∴ Total = 7 016 + 4 = 7 020.*

*Hence log 5.034 356 = 0.702 0 (since 5.034 lies between 1 and 10).*

## Exercise 8.8

Use 4-figure tables to find the logarithm of

1. (a) 5.372
   (b) 4.185
   (c) 7.321
2. (a) 3.712
   (b) 4.713
   (c) 8.495

3. (a) 7.430
   (b) 2.49
   (c) 6.213
4. (a) 3.161 4
   (b) 3.214 3
   (c) 4.111 6

## Logarithms of numbers greater than 10

We already know that log 10 = 1, log 100 = 2, log 1 000 = 3, and so on.

We can then, see that if a number is greater than 10, its logarithm must be greater than 1. The logarithm must, therefore, contain an integral part (whole number) and a decimal part. The integral part is called the **characteristic** and the decimal part the **mantissa** .

The logarithm of a number greater than 10 is obtained as follows.

1. Write the number in standard form, i.e. as $a \times 10^{n}$, e.g. log 535.7 = log $5.357 \times 10^2$ .
2. Using tables, find the logarithm of $a$, e.g. log 5.357 = 0.729 0
3. By inspection, state the logarithm of $10^{n}$, e.g. log $10^2 = 2$
4. Add the powers to get logarithm of the number, e.g. $10^{0.729\,0} \times 10^2 = 10^{2.729\,0}$ .

   ∴ log 535.7 = 2.729 0

***Example 8.15***

*Find the logarithm of*

*(a) 20.73*

*(b) 30 550*

***Solution***

| Number | Standard form | As power of 10 | Logarithm of number |
|---|---|---|---|
| (a) 20.73 | $2.073 \times 10^1$ | from tables $10^{0.3166} \times 10^1$ $= 10^{1 + 0.3166}$ $= 10^{1.3166}$ | Characteristic Mantissa 1.316 6 |
| (b) 30 550 | $3.055 \times 10^4$ | from tables $10^{0.4850} \times 10^4$ $= 10^{0.4850 + 4}$ $= 10^{4.4850}$ | 4.485 0 |

## Exercise 8.9

1. Express in standard form
   - (a) 973
   - (b) 48.6
   - (c) 5 000
   - (d) 5 270 000
   - (e) 4 850
   - (f) 72 536
2. State the characteristic of the logarithm of
   - (a) 15
   - (b) 4 800
   - (c) 600
   - (d) 2
   - (e) 93 000 000
   - (f) 39.8
   - (g) 1 250
   - (h) 23.4
   - (i) 735.6
3. Use 4-figure tables to find the logarithm of
   - (a) 3.16
   - (b) 31.6
   - (c) 316
   - (d) 3 160

(e) 31 600
(f) 316 000

What do you notice about the logarithms of these numbers?

4. Find the logarithm of

(a) 247
(b) 3 600
(c) 10 900
(d) 15.7
(e) 9 402
(f) 15 455

5. Write the following numbers as powers of 10.

(a) 7 863
(b) 12.41
(c) 428.45
(d) 98 370
(e) 91.93
(f) 108 341
(g) 128
(h) 25.997
(i) 368 701

6. State the logarithms of the numbers below.

(a) $10^{0.863\,5}$
(b) $10^{1.893\,4}$
(c) $10^{2.779\,1}$
(d) $10^{4.891\,1}$
(e) $10^{5.607\,1}$
(f) $10^{7.301\,0}$

## Logarithms of numbers between 0 and 1

Consider the number 0.064 8.

$$0.064\,8 = 6.48 \times 10^{-2}$$

From tables, log 6.48 = 0.811 6

$$0.064\,8 = 10^{0}\,.^{81}\,1\,6 \times 10^{-}\,2$$

$$= 10^{-}\,2 + 0.811\,6$$

Thus, $\log 0.064\,8 = \underbrace{^{-}2}_{\text{Negative characteristic}} + \underbrace{0.811\,6}_{\text{Positive mantissa}}$

Such logarithm as $^{-}2 + 0.811\,6$ is usually written as $\mathbf{\bar{2}.811\,6}$ to show that only 2 is negative.

$\bar{2}.811\,6$ is read as '*bar two point eight one one six* '.

Notice that if a number lies between zero and 1, its logarithm consists of two parts, a negative characteristic and a positive mantissa. Such a logarithm is written with the **minus sign above the characteristic** to show that only the characteristic is negative.

### *Example 8.16*

*Find the logarithm of 0.003 681.*

### *Solution*

| *Number* | *Standard form* | *As power of 10* | *Logarithm* |
|---|---|---|---|
| *0.003 681* | $3.681 \times 10^{-3}$ | $10^{0.565\,9} \times 10^{-3}$<br>$= 10^{-3+0.565\,9}$<br>$= 10^{\bar{3}.565\,9}$ | $\bar{3}.565\,9$ |

Note that logarithms of negative numbers do not exist, e.g. log ($^{-}$ 814 5) does not exist.

## Exercise 8.10

1. Express in standard form:
   - (a) 0.091
   - (b) 0.008 1
   - (c) 0.673
   - (d) 0.002 8
2. State the characteristic of the logarithm of
   - (a) 0.48

(b) 0.008 914
(c) 0.000 003 482
(d) 0.069 35

3. Find the logarithm of

(a) 0.763 815
(b) 0.008 413
(c) 0.104 4
(d) 0.000 987
(e) 0.342
(f) 0.048 3
(g) 0.007 6
(h) 0.902
(i) 0.045 034
(j) 0.401 3
(k) 0.007 137 9
(l) $4.03 \times 10^{-5}$
(m) $6.104 \times 10^{-6}$
(n) $4.28 \times 10^{-10}$

4. Copy and complete Table 8.4 .

| Number | Standard form | Product of powers of 10 | Logarithm |
|---|---|---|---|
| 2.073 | $2.073 \times 10^0$ | $10^{0.3166} \times 10^0$ | 0.316 6 |
| 0.207 3 | $2.073 \times 10^{-1}$ | $10^{0.3166} \times 10^{-1}$ | $\bar{1}.3166$ |
| 0.020 73 | | | |
| 0.002 073 | | | |
| 0.000 207 3 | | | |
| 0.000 020 73 | | | |

*Table 8.4*

What do you notice about the logarithms of the numbers in Table 8.4? Give a reason for your observation.

5. Express the following as powers of 10.

(a) 0.053 46
(b) 0.896 3
(c) 0.996 3

(d) 0.873 6
(e) 0.004 89
(f) 0.000 462 1

6. State the logarithms of the numbers below.

(a) 100.653 8
(b) 101.986 3
(c) $10^{\bar{2}.753\,2}$
(d) $10^{\bar{3}.869\,4}$

## Antilogarithms

Given the logarithm of a number, we can find the corresponding number by using **Tables of antilogarithms** . This means reversing the process of finding the logarithm of a number.

### *Example 8.17*

*Find the value of $10^{0.707\,6}$ .*

### *Solution*

*Since the power of 10 represents the logarithm of a number, then the logarithm of the required number is 0.707 6.*

*$\therefore$ log x = 0.707 6 where x is the required number.*

*Since 0.707 6 lies between 0 and 1, x must lie between 1 and 10.*

*Using antilogarithm tables (Table 8.5 ), look for 0.70 down the first column.*

***Antilogarithms***

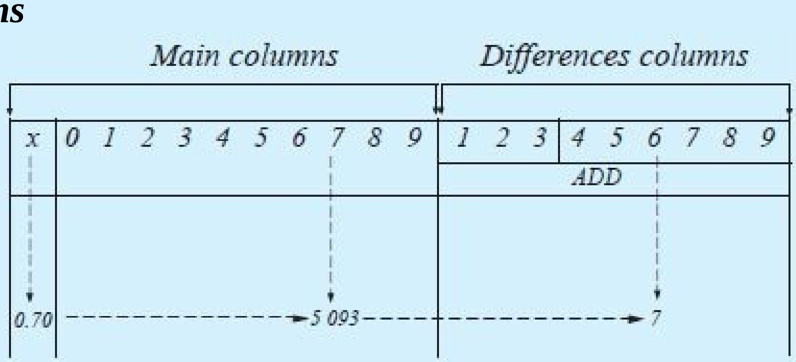



*Table 8.5*

*Move across the page along row 0.70 until you get to the column headed 7 in* *the main columns.*

*The number at the intersection of the row and the column is 5 093. Note it down.*

*Continue across the page along the same row until you get to the differences column headed 6. The number at this intersection is 7. This 7 must be added to 5 093 and we get 5 100. Therefore, the number whose logarithm is 0.707 6 is 5.1.*

*i.e.* $\log x = 0.707\ 6$

$\therefore x = 5.1$

## Example 8.18

*Use antilogarithm tables to find the number whose logarithm is*

(a) $2.713\ 6$

(b) $\bar{3}.873\ 8$

## Solution

(a) *Let a be the number whose logarithm is 2.713 6.*

$\therefore \log a = 2.713\ 6$

$\Rightarrow a = 10^{2.713\ 6}$

$= 10^2 \times 10^{0.713\ 6}$

*Now using antilogarithm tables, Row 0.71 and column 3 give 5 164. Differences column 6 gives 7.*

*The total is* $5\ 164 + 7 = 5\ 171$

$\therefore 10^{0.713\ 6} = 5.171$

*Thus,* $a = 5.171 \times 10^2$

$= 517.1$

(b) *Let b be the number whose logarithm is* $\bar{3}.873\ 8$

$\therefore \log b = \bar{3}.873\ 8$

$\Rightarrow b = 10^{\bar{3}.873\ 8}$

$= 10^{-3} \times 10^{0.873\ 8}$

*Using antilogarithm tables, Row 0.87 and column 3 give 7 464.*

*Differences column 8 gives 14 The total is* $7\ 464 + 14 = 7\ 478$

$\therefore 10^{0.873\,8} = 7.478$

Thus, $b = 7.478 \times 10^{-3}$

$= 0.007\,478$

## Exercise 8.11

Use antilogarithm tables to find the numbers whose logarithms are given in Questions 1 to 5.

**1.** (a) 0.481 5

(b) 2.307 1

(c) 1.811 4

**2.** (a) 3.042 1

(b) 2.116 3

(c) 0.947 2

**3.** (a) 0.832 1

(b) 4.673 9

(c) 1.816 4

**4.** (a) $\bar{1}.621\,3$

(b) $\bar{1}.814\,4$

(c) $\bar{3}.411\,8$

**5.** (a) $\bar{1}.651\,1$

(b) $\bar{2}.495\,3$

(c) $\bar{4}.688\,3$

valuate the following:

**6.** (a) $10^{0.483\,5}$

(b) $10^{1.704\,0}$

(c) $10^{2.780\,1}$

**7.** (a) $10^{\bar{1}.905\,2}$

(b) $10^{\bar{2}.525\,5}$

(c) $10^{\bar{3}.660\,3}$

**8.** (a) $10^{\bar{4}.199\,8}$

(b) $10^{\bar{2}.071\,6}$

(c) $10^{\bar{1}.077\,9}$

Find the numbers whose logarithms are given in Questions 9 to 11.

**9.** (a) $^{-}1 + 0.592\ 2$

(b) $^{-}2 + 0.8645$

**10.** (a) $^{-}1 + 0.477\ 1$

(b) $^{-}2 + 0.608\ 5$

**11.** (a) $^{-}4 + 0.908\ 5$

(b) $^{-}2 + 0.301\ 6$

## Using calculators to find logarithms and antilogarithms

As with mathematical tables, we can use a calculator to perform computations involving logarithms and antilogarithms. We learnt that to find the number whose logarithm is given means reversing the process of finding logarithms. We can do the same on a calculator by using either $\boxed{10^x}$ , i.e. $\boxed{\text{INV}}$ $\boxed{\log}$ , or key in 10, press $\boxed{x^y}$ and key in the given value.

***Example 8.19***

*Use a calculator to find*

*(a) the logarithm of (i) 28.95 (ii) 0.008 615*

*(b) the number whose logarithm is 0.267 8.*

***Solution***

*(a) (i) To find the logarithm of 28.95, follow the sequence:*

*Press* $\boxed{\log}$ *, key in 28.95, press =* $\boxed{=}$ *.*

*Answer: 1.461 648 568.*

- *(ii) Press* $\boxed{\log}$ *, key in 0.008 615, press* $\boxed{=}$ *.*
  *Answer: –2.064 744 718.*

***Note** that with regard to logarithm of a number less than 1, the calculator does not separate the characteristic from the mantissa as we do when using logarithm tables. However, should it be necessary to do so, you should be able to convert this logarithm to bar form.*

*For example, given log x =* $^{-}2.06$*, we convert it as follows:*

$\log x = {}^{-}2 + {}^{-}0.06$

$= ({}^{-}1 + {}^{-}2) + ({}^{-}0.06 + 1)$ *(to have a positive mantissa)*

$= \bar{3}.94$

*(b) Let the number whose logarithm is 0.267 8 be x.*

$\log_{10} x = 0.267\ 8$

$\therefore x = 10^{0.267\ 8}$

*Key in 10, press* $\boxed{x^y}$ *, key in 0.267 8. press* $\boxed{=}$ .

*The answer is 1.852 678 823 7.*

**Important point:**

You may have found out that with some calculators, the sequence in Example 8.19 does not work. This happens because different calculators are programmed differently. It is important, therefore, that you study the manual of your calculator so that you know how it works.

## Exercise 8.12

Use a calculator to evaluate the following:

1. Find the logarithm of
   - (a) 8.296
   - (b) 6.788 2
   - (c) 3.796 84
   - (d) 174
   - (e) 9 118
   - (f) 11 390
   - (g) 0.047 8
   - (h) 0.000 461
   - (i) $1.603 \times 10^{-4}$
2. Find the numbers whose logarithms are
   - (a) 0.846 3
   - (b) 5.276 3
   - (c) $\bar{2}.718\ 6$
   - (d) $\bar{1}.143\ 7$

3. Evaluate the following:

(a) $10^{0.042\,1}$

(b) $10^{3.280\,4}$

(c) $10^{\bar{1}.423\,2}$

(d) $10^{\bar{4}.734\,3}$

## Using logarithms in multiplication

When multiplying numbers, we could use logarithms. Usually, we use logarithms to avoid long and cumbersome multiplications, and when accuracy is required only to a limited number of significant figures, e.g. 3 or 4 s.f.

It is possible to use logarithms in multiplication because, as we have already seen, any number can be written as a power of 10, and we know that the power or index is the logarithm of that number. Then, using multiplication law of indices, we get the logarithm of the product and find its antilogarithm, i.e.

Let $a = 10^x$ and $b = 10^y$

Then $a \times b = 10^x \times 10^y = 10^{x+y}$

Now, $\log a = x$, $\log b = y$, and

$\log(a \times b) = \log 10^{x+y} = x + y$

Thus, $\log(a \times b) = \log a + \log b$

So, to get the product $a \times b$, get the sum of logs and find the antilog.

### *Example 8.20*

*Use logarithms to evaluate*

*(a) 3.742 4 × 41.68*

*(b) 91.85 × 3.467 × 125.3*

### *Solution*

(a)

| Number | Standard form | Logarithm |
|---|---|---|
| 3.742 | $3.742 \times 10^0$ | 0.573 1 + |
| 41.68 | $4.168 \times 10^1$ | 1.619 9 |
| | $10^{2.193\,0}$ ← | 2.193 0 |
| | $= 10^2 \times 10^{0.193}$ | |
| 156 ← | $= 10^2 \times 1.560$ | |

$\therefore$ 3.742 × 41.68 = 156 correct to 3 s.f.

(b)

| Number | Standard form | Logarithm |
|---|---|---|
| 91.85 | $9.185 \times 10^1$ | 1.963 0 |
| 3.467 | $3.467 \times 10^0$ | 0.540 0 + |
| 125.3 | $1.253 \times 10^2$ | 2.097 9 |
| | $10^{4.600\,9}$ ← | 4.600 9 |
| | $= 10^4 \times 10^{0.600\,9}$ | |
| 39 890 ← | $= 10^4 \times 3.989$ | |

$\therefore$ 91.85 × 3.467 × 125.3 = 39 890

To find the product of two numbers, add their logarithms. Then find the antilogarithm of the sum, i.e. $\log (x \times y) = \log x + \log y$ since if $\log x = a$, and $\log y = b$, then

$$xy = 10^a \times 10^b = 10^{(a\,+\,b)}$$

Then find the antilogarithm of $(a + b)$ .

## Exercise 8.13

Use 4-figure logarithm tables to work out the following.

**1.** (a) 4.8 × 6.4
  (b) 23.66 × 14.21

**2.** (a) 3.2 × 17.41
  (b) 16.82 × 72.13

**3.** (a) 49.2 × 2.345
  (b) 3.081 × 17.26

**4.** (a) 42.44 × 53.81
  (b) 17.6 × 2.48

**5.** (a) $3.481 \times 2.673$

(b) $23.12 \times 42.67$

**6.** (a) $3.731 \times 21.42$

(b) $74.01 \times 2.335$

**7.** (a) $6.71 \times 42.62$

(b) $9.92 \times 4.59$

**8.** (a) $25.9 \times 96.3$

(b) $451.9 \times 637.9$

**9.** $2.041 \times 5.763 \times 241.2$

**10.** $8.743 \times 763.1 \times 42.68$

**11.** $563.7 \times 26.14 \times 3.211$

**12.** $546.6 \times 9\,748 \times 4.385$

## Using logarithms in division

To divide one number by another, we subtract the logarithm of the denominator from the logarithm of the numerator (division law of indices), then we find the antilogarithm of the difference, i.e. just as

$$10x \div 10^{y} = 10^{x-y},$$

then $\log (x \div y) = \log x - \log y$.

### *Example 8.21*

Use 4-figure logarithm tables to find the value of $35.82 \div 14.21$.

### *Solution*

| *Number* | *Standard form* | *Logarithm* |
| --- | --- | --- |
| *35.82* → | *$3.582 \times 10^1$* → | *1.554 1* – |
| *14.21* → | *$1.421 \times 10^1$* → | *1.152 6* |
| | *$10^{0.401\,5}$* ← | *0.401 5* |
| *2.521* ← | *$= 2.521 \times 10^0$* | |

$\therefore \; 35.82 \div 14.21 = 2.521$

## Exercise 8.14

Use 4-figure logarithm tables to find the value of the following.

**1.** (a) $24.2 \div 16.1$

(b) $327 \div 1.53$

**2.** (a) $48.12 \div 6.123$

(b) $78.12 \div 12.64$

**3.** (a) $812.1 \div 16.52$

(b) $218.4 \div 17.3$

**4.** (a) $512 \div 6.43$

(b) $17.21 \div 14.18$

**5.** (a) $653.1 \div 37.24$

(b) $518.6 \div 173.4$

**6.** (a) $83.41 \div 15.76$

(b) $92.86 \div 68.93$

## Bar numbers

Earlier in this chapter, we wrote the logarithms of numbers between 0 and 1 with a minus sign above the characteristic. When written in this form, the logarithms may then be referred to as **bar numbers** .

When we divide or multiply numbers or find powers or roots of numbers, we sometimes end up with such numbers. To be able to deal with such logarithms, we need to be able to perform arithmetic operations on them.

## Adding bar numbers

***Example 8.22***

*Simplify (a)* $\bar{2}.89 + \bar{5}.47$ *(b)* $\bar{3}.76 + \bar{4}.85$

***Solution***

(a) $\bar{2}.89 + \bar{5}.47$

$$\begin{array}{rl} = & \bar{2} + 0.89 + \\ & \underline{\bar{5} + 0.47} \\ & \bar{7} + 1.36 \end{array}$$

(Write the numbers vertically, separating the integral parts from the decimal parts and add).

$= \bar{7} + 1 + 0.36$

$= \bar{6} + 0.36$

(Simplify the integral parts, i.e. $^-7 + 1 = {^-6}$).

$= \bar{6}.36$ (Write as a single number in bar form).

(b) $\bar{3}.76 + 4.85$

$$\begin{array}{rl} = & \bar{3} + 0.76 + \\ & \underline{4 + 0.85} \\ & 1 + 1.61 \\ = & 2.61 \end{array}$$

## Subtracting bar numbers

### Example 8.23

Simplify (a) $\bar{4}.21 - 1.73$ (b) $\bar{2}.36 - \bar{6}.87$

### Solution

(a) $\bar{4}.21 - 1.73$

$$\begin{array}{rl} = & \overset{^-1}{\bar{4}} + \overset{^+1}{0.21} - \\ & \underline{1 + 0.73} \\ & {^-6} + 0.48 \\ = & \bar{6}.48 \end{array}$$

(Borrow 1 from $^-4$ to leave $^-5$)

($^-5 - 1 = {^-6}$)

(b) $\bar{2}.36 - \bar{6}.87$

$$\begin{array}{rl} = & \overset{^-1}{\bar{2}} + \overset{^+1}{0.36} - \\ & \underline{\bar{6} + 0.87} \\ & 3 + 0.49 \\ = & 3.49 \end{array}$$

(Borrow 1 from $^-2$ to leave $^-3$)

(subtract $^-6$ from $^-3$) (i.e. $^-3 - (^-6) = {^-3} + 6 = 3$)

## Exercise 8.15

Calculate the following.

**1.** (a) $\bar{1}.2 + \bar{3}.7$

(b) $\bar{3}.2 + \bar{2}.4$

(c) $\bar{2}.3+3.4$

**2.** (a) $\bar{1}.3+2.8$

(b) $\bar{4}.8+2.8$

(c) $\bar{1}.6+\bar{2}.7$

**3.** (a) $\bar{3}.6-2.4$

(b) $2.7-\bar{3}.5$

(c) $4.1-\bar{2}.7$

**4.** (a) $\bar{1}.6-\bar{2}.3$

(b) $\bar{3}.4-\bar{4}.2$

(c) $\bar{2}.3-1.6$

## Division of bar numbers

***Example 8.24***

*Evaluate* $\bar{7}.42 \div 2$

***Solution***

*To divide a bar number, the characteristic must be exactly divisible by the divisor. If not, we subtract the least positive number which makes it so.*

*Thus* $\bar{7}.42 \div 2 = ({}^{-}7 + 0.42) \div 2$

$= ({}^{-}7 - 1 + 1 + 0.42) \div 2$

*(Subtract 1 from the characteristic and add to the mantissa so that the value of the number does not change)*

$= \dfrac{{}^{-}8 + 1.42}{2}$

$= \dfrac{{}^{-}8}{2} + \dfrac{1.42}{2}$

$= {}^{-}4 + 0.71$

$= \bar{4}.71$

## Multiplication of bar numbers

***Example 8.25***

*Simplify* $\bar{3}.6 \times {}^{-}2$

### *Solution*

$$
\begin{aligned}
\bar{3}.6 \times {}^{-}2 &= ({}^{-}3 + 0.6) \times {}^{-}2 \\
&= {}^{-}3 \times {}^{-}2 + 0.6 \times {}^{-}2 \\
&= 6 + {}^{-}1.2 \\
&= 4.8
\end{aligned}
$$

## Exercise 8.16

Evaluate the following.

**1.** (a) $\bar{2}.12 \times 8$

(b) $\bar{4}.78 \times {}^{-}2$

**2.** (a) $\bar{5}.264 \div 2$

(b) $\bar{2}.714 \div 3$

**3.** (a) $\bar{4}.62 \times {}^{-}3$

(b) $\bar{2}.913 \times {}^{-}4$

**4.** (a) $\bar{3}.722\,2 \div {}^{-}2$

(b) $\bar{6}.199\,7 \div 4$

## Using bar numbers in multiplication and division

### *Example 8.26*

*Calculate (a) 42.8 × 0.003 251*

*(b) 0.356 ÷ 0.015 38*

### *Solution*

(a)

| Number | Log |
|---|---|
| 42.8 → | 1.631 4 |
| 0.003 251 → | $\bar{3}$.512 0 + |
| 0.139 1 ← | $\bar{1}$.143 4 |

$\therefore\ 42.8 \times 0.003\ 251 = 0.139\ 1$

(b)

| Number | Log |
|---|---|
| 0.356 → | $\bar{1}$.551 4 |
| 0.015 38 → | $\bar{2}$.186 9 − |
| 23.15 ← | 1.364 5 |

$\therefore\ 0.356 \div 0.015\ 38 = 23.15$

## Example 8.27

Calculate $\dfrac{2.61 \times 21.83 \times 0.073}{61.72 \times 11.73}$

## Solution

| Number | Log | |
|---|---|---|
| 2.61 → | 0.416 6 | Evaluate logarithm of numerator |
| 21.83 → | 1.339 1 + | |
| 0.073 → | $\bar{2}$.863 3 | |
| Nume. | 0.619 0 | |
| 61.72 → | 1.790 4 + | Evaluate logarithm of denominator and subtract log of denominator from log of numerator |
| 11.73 → | 1.069 3 | |
| Denom. | 2.859 7 | |
| 0.005 745 ← | $\bar{3}$.759 3 | |

$$\therefore\ \frac{2.61 \times 21.83 \times 0.073}{61.72 \times 11.73} = 0.005\ 745$$

## Exercise 8.17

Use logarithm tables to calculate

1. 543.2 × 0.562 1
2. 8.743 × 763.1 × 0.426 8
3. 456.6 × 974.8 × 0.003 485
4. 563.7 × 26.14 × 0.032 1

5. $342.2 \div 4.244$
6. $0.621\ 3 \div 2.842$
7. $0.834\ 1 \div 15.76$
8. $596.6 \div 2\ 432$
9. Calculate
   (a) PQ
   (b) Q ÷ P
   (c) P ÷ Q
   if P = 37.32 and Q = 46.54.
10. $\frac{5.4 \times 0.006\ 4}{0.086}$
11. $\frac{0.48}{0.73 \times 0.92}$
12. $\frac{0.872 \times 3\ 150 \times 0.034\ 5}{0.56 \times 0.93}$

## Powers and roots

### Powers

Raising a number to a power corresponds to multiplication of the logarithm of the number by the index.

***Example 8.28***

*Calculate $29.11^2$ correct to 4 s.f.*

***Solution***

*Method 1*

| Number | Log |
|---|---|
| 29.11 | 1.464 0 + |
| 29.11 | 1.464 0 |
| 847.2 | 2.928 0 |

*Or*

*Method 2*

| Number | Log |
|---|---|
| $29.11^2$ | $1.464\ 0 \times 2$ |
| 847.2 | 2.928 0 |

$$\therefore 29.11^2 = 847.2$$

***Example 8.29***

*Find the value of (0.849 5)$^4$ .*

***Solution***

*Method 1*

| Number | Log |
|---|---|
| 0.849 5 → | 1.929 2 |
| 0.849 5 → | 1.929 2 |
| 0.849 5 → | 1.929 2 |
| 0.849 5 → | 1.929 2 |
| 0.521 0 ← | 1.716 8 |

*Method 2*

| Number | Log |
|---|---|
| 0.849 $5^4$ → | 1.929 2 × 4 |
| 0.521 0 ← | 1.716 8 |

$\therefore 0.849\ 5^4 = 0.521\ 0$

Method 1 is only suitable for small powers, i.e. squares and cubes. For any other power, including fractional powers, method 2 is more appropriate to use.

In general, $\log x^n = n \times \log x$, written simply as $n \log x$ for all values of $n$.

## Roots

The inverse process to raising a number to a power is taking the corresponding root.

To find the root of a number, we divide the logarithm of the number by the order of the root.

***Example 8.30***

*Calculate the cube root of 35.64.*

***Solution***

*Remember* $\sqrt[3]{35.64} = 35.64^{\frac{1}{3}}$

$\therefore \log 35.64^{\frac{1}{3}} = \frac{1}{3}\log 35.64$

| Number | Log |
|---|---|
| $35.64^{\frac{1}{3}}$ → | $1.551\,9 \times \frac{1}{3}$ |
| $3.291$ ← | $0.517\,3$ |

$\therefore \sqrt[3]{35.64} = 3.291$ (correct to 4 s.f.)

In general, $\log \sqrt[n]{x} = \log x^{\frac{1}{n}}$
$= \frac{1}{n}\log x$

## Exercise 8.18

Use logarithms to evaluate

**1.** (a) $17.3^2$
(b) $1.312^5$
(c) $2.312^4$

**2.** (a) $2.812^3$
(b) $1.517^4$
(c) $2.723^5$

**3.** (a) $17.41^{\frac{2}{3}}$
(b) $82.7^{\frac{1}{3}}$
(c) $2.971^2$

**4.** (a) $34^3$
(b) $2.24^5$
(c) $\sqrt{1.62}$

**5.** (a) $\sqrt{55}$
(b) $\sqrt[4]{22.5}$
(c) $\sqrt[3]{181.2}$

**6.** (a) $\sqrt[3]{28.41}$
(b) $\sqrt[4]{3\,100}$
(c) $\sqrt{8\,910}$

**7.** (a) $\sqrt[3]{3.765}$
(b) $\sqrt[3]{786.5}$

(c) $\sqrt[4]{25}$

## Combined operations

### Example 8.31

*Find the value of*

$$\sqrt{\frac{14.73 \times 22.41}{82.3}}$$

### Solution

### Example 8.32

| Number | Log |
|---|---|
| 14.73 | 1.168 2 + |
| 22.41 | 1.350 4 |
| | 2.518 6 − |
| 82.3 | 1.915 4 |
| | 0.603 2 ÷ 2 |
| 2.003 | 0.301 6 |

$$\therefore \sqrt{\frac{14.73 \times 22.41}{82.3}} = 2.003$$

### Solution

$$\frac{\sqrt{48\,700} \times 8.93}{3.142^2 \times 5.67}$$

### Solution

| Number | Log | |
|---|---|---|
| $\sqrt{48\ 700}$ | $4.687\ 5 \times \frac{1}{2}$ | |
| | $= 2.343\ 8 +$ | Evaluate and simplify logarithm of numerator |
| $8.93$ | $0.950\ 9$ | |
| | $3.294\ 7$ | Evaluate and simplify logarithm of denominator and subtract it from log of numerator |
| $3.142^2$ | $0.497\ 2 \times 2$ | |
| | $= 0.994\ 4 +$ | |
| $5.67$ | $0.753\ 6$ | |
| | $1.748\ 0$ | |
| $35.22$ | $1.546\ 7$ | |

$$\therefore \quad \frac{\sqrt{48\ 700} \times 8.93}{3.142^2 \times 5.67} = 35.22 \ \ (4 \text{ s.f.})$$

## *Example 8.33*

*Evaluate using logarithms:*

$$\frac{\sqrt{0.843\ 2} - \sqrt[3]{0.752\ 6}}{\sqrt{0.843\ 2} + \sqrt[3]{0.752\ 6}}$$

## *Solution*

*Remember that logarithms are not used for addition or subtraction.*

1. *Calculate* $\sqrt{0.843\ 2}$

| Number | Log |
|---|---|
| $\sqrt{0.843\ 2}$ | $\bar{1}.925\ 9 \div 2$ |
| $0.918\ 1$ | $\bar{1}.962\ 9$ |

2. *Calculate* $\sqrt[3]{0.752\ 6}$

| Number | Log |
|---|---|
| $\sqrt[3]{0.752\ 6}$ | $\bar{1}.876\ 5 \div 3$ |
| $0.909\ 5$ | $\bar{1}.958\ 8$ |

3. *Now, calculate the difference in the numerator, and the sum in the denominator.*

$$\frac{\sqrt{0.843\,2}-\sqrt[3]{0.752\,6}}{\sqrt{0.843\,2}+\sqrt[3]{0.7526}} = \frac{0.918\,1-0.909\,5}{0.918\,1+0.909\,5}$$

$$= \frac{0.008\,6}{1.827\,6}$$

| Number | Log |
|---|---|
| 0.008 6 → | $\bar{3}.934\,5$ – |
| 1.827 6 → | 0.262 0 |
| 0.004 814 ← | $\bar{3}.672\,5$ |

$$\therefore \quad \frac{\sqrt{0.843\,2}-\sqrt[3]{0.752\,6}}{\sqrt{0843\,2}+\sqrt[3]{0.752\,6}} = 0.004\,704$$

## Exercise 8.19

1. Use logarithms to evaluate the following expressions.

   (a) $\frac{22.67-11.43}{22.67+11.43}$

   (b) $\sqrt{\frac{0.003}{0.017\,26}}$

   (c) $\sqrt[3]{0.828}$

   (d) $\frac{0.671^3 \times 0.042^2}{\sqrt{0.061}}$

   (e) $\frac{\sqrt{1.1} \times 14.23^2}{39.67}$

   (f) $\frac{8.072 \times \sqrt{0.743\,2}}{0.824^2}$

   (g) $\sqrt[5]{\frac{17.26}{43.81}}$

   (h) $\sqrt{\frac{82.41 \times 76.62}{7.389}}$

   (i) $\left(\frac{4.12 \times 71.93}{1.458 \times 82.44}\right)^{\frac{1}{3}}$

   (j) $\frac{48\sqrt{592}}{9.761}$

2. Given that log 3 = 0.477 1 and log 5 = 0.699 0, without further use of logarithm tables, find

(a) log $3^2$
(b) log $5^3$
(c) log 15
(d) log 45
(e) log 30
(f) log 50

3. If log $x$ = 2.634 8 and log $y$ = 3.516 3, find the logarithm of
   (a) $xy$
   (b) $x \div y$
   (c) $x^2 y$
   (d) $xy^2$
   (e) $\sqrt{xy}$
   (f) $\sqrt[3]{\frac{x}{y}}$

4. Find the logarithms of the following numbers to the indicated base (For example: $49 = 7^2$ . ∴ log7 49 = 2).
   (a) 81 to base 9
   (b) 64 to base 4
   (c) 25 to base 5
   (d) 16 to base 2
   (e) 27 to base 3
   (f) 216 to base 6

# 9 SIMILARITY I

## Introduction

In every day life, we encounter and talk about objects which have the same shape. In mathematics, such objects are said to be similar. Whenever we look at a house plan, make a scale model of a car, etc., we are making use of similar shapes.

Note that in every day use, the word ‘similar’ often means ‘roughly the same shape’. For example, the leaves coming from the same tree have roughly the same shapes, although it may be difficult to find two leaves which have the same shape or size. Likewise, all sheep are similar, but a shepherd will be able to distinguish between the sheep in his flock since no two sheep are exactly the same.

In mathematics, however, the word ‘similar’ is used to mean ‘**exactly the same shape** ’. For example, the shape of a picture on a television screen is exactly the same irrespective of the size of the screen. In this case, the pictures are similar.

## Similar objects

Consider Fig. 9.1(a) and (b) .

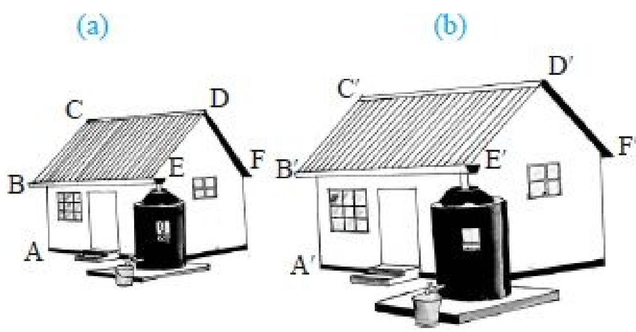



*Fig. 9.1*

The two drawings are exactly the same except for their sizes.

Measure the lengths of the corresponding lines BC and B′C′, CD and C′D′, DE and D′E′, DF and D′F′, etc.

Copy and complete the following.

$\frac{B'C'}{BC} =$ $\frac{C'D'}{CD} =$

$\frac{D'E'}{DE} =$ $\frac{D'F'}{DF} =$

What do you notice about the ratio of corresponding sides?

Measure the corresponding angles ∠A and ∠A′, ∠B and ∠B′, ∠C and ∠C′, etc.

What do you notice?

You should notice that the ratio of the lengths of corresponding lines is constant and that the corresponding angles are equal.

We say that the two drawings in Fig. 9.1 are **similar** .

Two solids are similar if

1. the ratio of the lengths of their corresponding sides is constant, and
2. the corresponding angles are equal.

***Example 9.1***

*A jewel box, of length 30 cm, is similar to a matchbox. If the matchbox is 5 cm long, 3.5 cm wide and 1.5 cm high, find the breadth and height of the jewel box.*

***Solution***

*The ratio of the lengths of the jewel box and the matchbox is*

$$\frac{\text{Length of jewel box}}{\text{Length of match box}} = \frac{30\text{ cm}}{5\text{ cm}} = 6$$

*This means that each edge of the jewel box is 6 times the length of the corresponding edge of the matchbox.*

*width of jewel box = 6 × 3.5 cm = 21 cm, and height of jewel box = 6 × 1.5 cm = 9 cm.*

## Exercise 9.1

1. A scale model of a double-decker bus is 7.0 cm high and 15.4 cm long. If the bus is 4.2 m high, how long is it?
2. A cuboid has a height of 15 cm. It is similar to another cuboid which is

9 cm long, 5 cm wide and 10 cm high. Calculate the area of the base of the larger cuboid.

3. A water tank is in the shape of a cylinder radius 2 m and height 3 m. A similar tank has radius 1.5 m. Calculate the height of the smaller tank.
4. Write down the dimensions of any two cubes. Are the two cubes similar? Are all cubes similar? Are all cuboids similar?
5. A designer has two models of a particular car. The first model is 15 cm long, 7.5 cm wide and 5 cm high. The second model is 3.75 cm long, 1.70 cm wide and 1.25 cm high. He says that the two are ‘accurate scale models’. Explain whether or not his claim could be true.

## Similar shapes

Now consider the shapes in Fig. 9.2 . Which shapes in the figure are similar to (a) ? Why are the others not?

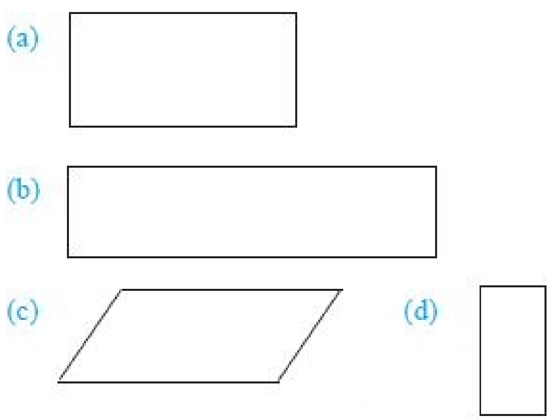



*Fig. 9.2*

We see that:

When two figures are similar,

1. the ratio of corresponding lengths is constant, **and**
2. corresponding angles are equal.

***Example 9.2***

*Determine whether the hexagons in Fig. 9.3 are similar. State your reasons.*

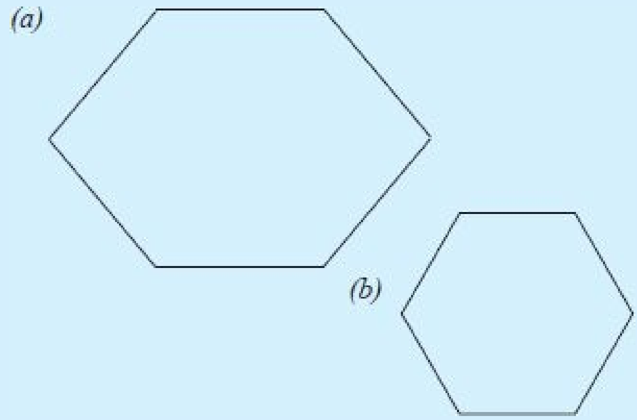



*Fig. 9.3*

***Solution***

*In Fig. 9.3 (a) , each side is 2 cm. In (b) , each side is 1.4 cm.*

*∴ Ratio of corresponding sides is* $\frac{1.4}{2}$ *= 0.7 (a constant)*

*In (a), there are two angles of 100°, and four angles of 130° each. Each angle in (b) is 120°.*

*∴ not all corresponding angles are equal.*

*Although the ratio of corresponding sides is constant, not all corresponding angles are equal.*
*Hence, the two hexagons are not similar.*

Remember that:

Two polygons are similar if

1. the ratio of corresponding sides is constant, **and**
2. the corresponding angles are equal.

## Construction of similar triangles

Draw a triangle with sides 4.6 cm, 3.0 cm and 3.4 cm. Draw another triangle whose sides are $1\frac{1}{2}$ times as long. What do you notice about the angles of the triangles? Are the two triangles similar?

Draw two triangles, one larger than the other, with angles of 55°, 75° and 50°. Are the two triangles similar?

We notice that:

Two **triangles** are similar if either

1. the ratio of corresponding sides is constant, **or**
2. the corresponding angles are equal.

Would the statement still be true if 'triangles' is replaced with 'polygons'? The answer is 'No': With all polygons other than triangles, the 'or' must be replaced with 'and'.

Further constructions involving similar figures will be done later in the section covering enlargement.

***Example 9.3***

*Fig. 9.4 shows two triangles ABC and PQR. Calculate the lengths BC and PQ.*

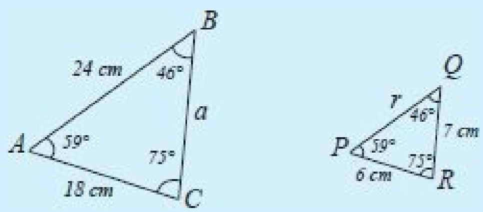



*Fig. 9.4*

***Solution***

*Since the corresponding angles are equal, Δs ABC and PQR are similar.*

∴ *the ratio of corresponding sides is constant.*

*Thus,* $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{AC}{PR}$

$$\text{i.e. } \frac{24}{r} = \frac{a}{7} = \frac{18}{6}$$

$$\Rightarrow \frac{a}{7} = \frac{18}{6} \text{ i.e. } \frac{a}{7} = 3$$

$$\therefore a = 21 \text{ i.e. } BC = 21 \text{ cm}$$

$$\text{Also, } \frac{24}{r} = \frac{18}{6} \text{ i.e. } \frac{24}{r} = 3$$

$$\Rightarrow 3r = 24$$

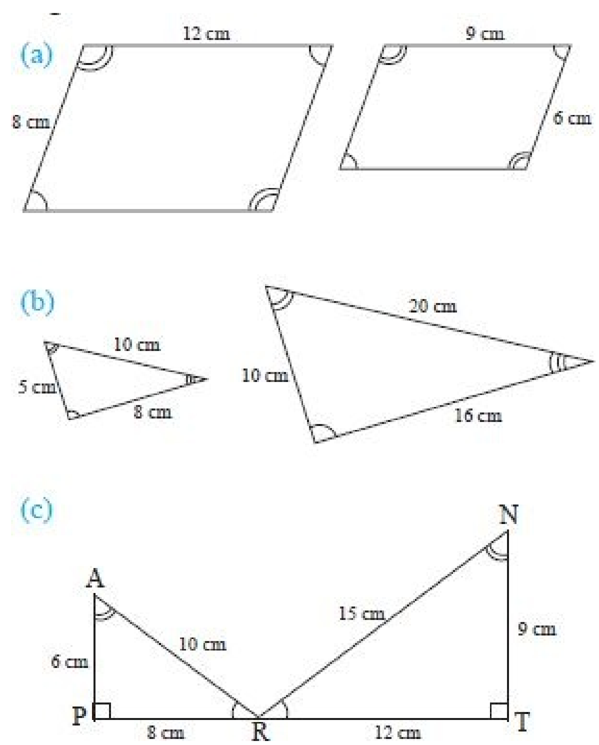
$$\therefore r = 8 \text{ i.e. } PQ = 8 \text{ cm}$$

**Note:**

1. The symbol ~ is used for similarity so that ΔABC ~ ΔPQR is read as 'Triangle ABC is similar to triangle PQR'.
2. Since the two triangles in Fig. 9.4 are alike in everything except size, it can be said that ΔPQR is a scale drawing of ΔABC with a scale 1 : 3.

## Exercise 9.2

1. State if, and why, the pairs of shapes in Fig. 9.5 are similar.



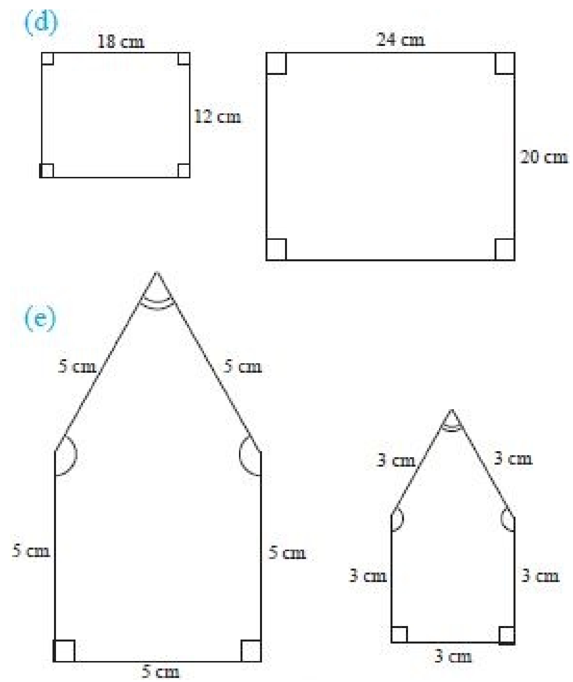



*Fig. 9.6*

2. The triangles in each pair in Fig. 9.6 are similar. Find $x$ .

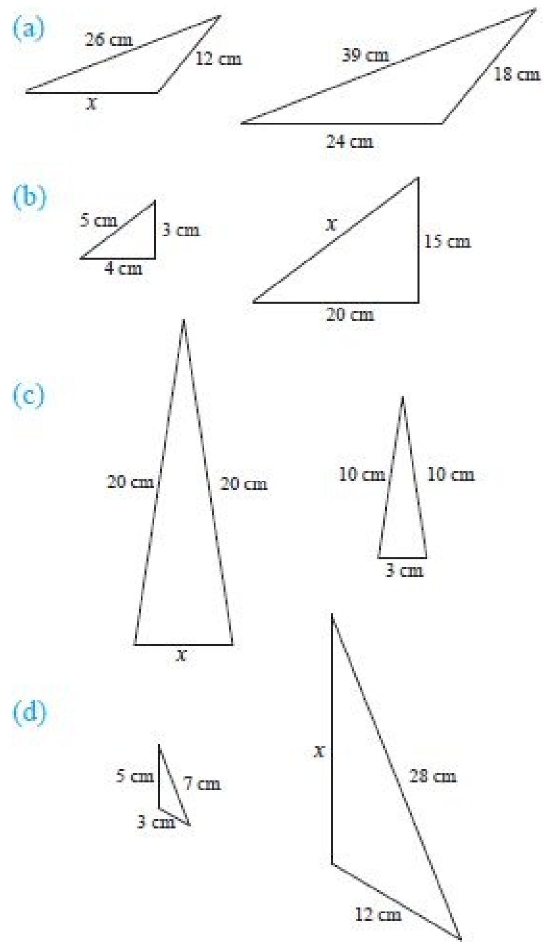



*Fig. 9.6*

3. Show that the two triangles in Fig. 9.7 are similar. Hence calculate AC and PQ.

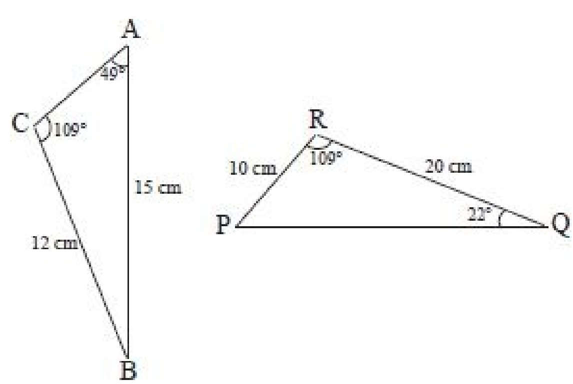



*Fig. 9.7*

4. Which two triangles in Fig. 9.8 are similar? State the reason. If AB = 6 cm, BC = 4 cm and DE = 9 cm, calculate BD.

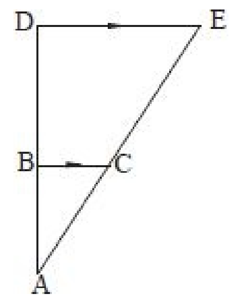



*Fig. 9.8*

5. Construct two triangles ABC and PQR with sides 3, 3, 5 cm and 12, 12, 20 cm respectively. Measure all the angles. Are triangles ABC and PQR similar? Are all isosceles triangles similar? Are all equilateral triangles similar?

6. The vertices of three right-angled triangles are given below:

   A(3, 3), B(4, 5), C(3, 5);

   P(1, 3), Q(1, 5), R(2, 4);

   X($^-$2, 3), Y(1, $^-$1), Z($^-$2, $^-$1).

   Which two triangles are similar?

7. Measure the length and breadth of this textbook. Measure also the length and breadth of the top of your desk. Are the two shapes similar? If not, make a scale drawing of a shape that is similar to the top of your desk.

8. A photograph which measures 27 cm by 15 cm is mounted on a piece of card so as to leave a border 2 cm wide all the way round. Is the shape of the card similar to that of the photograph? Give reasons for your answer.

9. In Fig. 9.9 , identify two similar triangles. Use the similar triangles to calculate $x$ and $y$ .

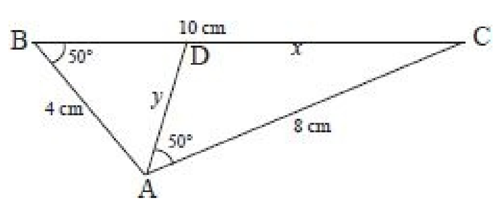



*Fig. 9.9*

10. Nkhata used similar triangles to find the distance across a river. To

construct the triangles he made the measurements shown in Fig. 9.10 . Find the distance across the river.

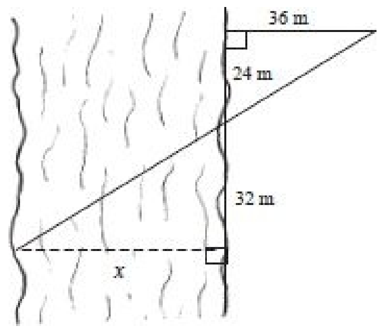



*Fig. 9.10*

## Enlargement and its properties

Compare the drawings in Fig. 9.11 . They are exactly alike except for the size. We say that the larger drawing (b) is an **enlargement** of the smaller one (a) .

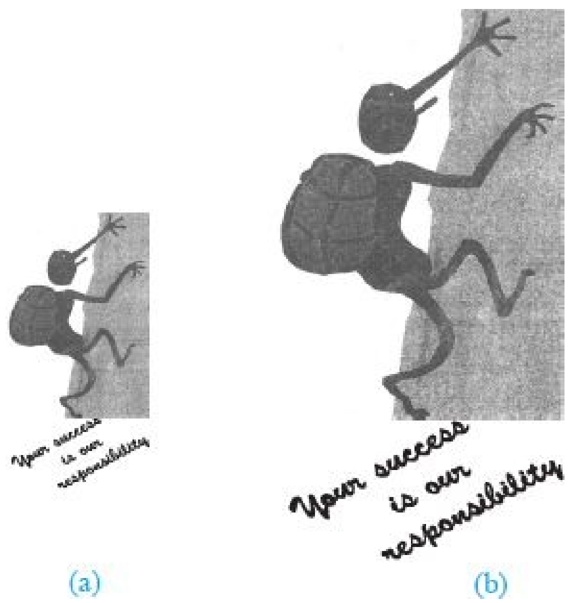



*Fig. 9.11*

If we regard (a) as the object, then (b) is the image under the transformation of **enlargement** .

Fig. 9.12 shows a triangle ABC and its image A′B′C′. Lines AA′, BB′ and CC′, produced, meet at a common point O. The point O is called the **centre of enlargement** . Δs ABC and A′B′C′ are **similar** . What is the ratio of lengths of corresponding sides?

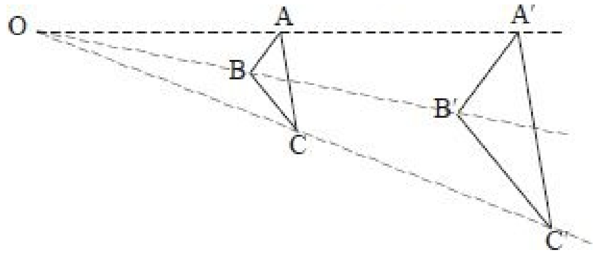



*Fig. 9.12*

By measuring, determine the value of $\frac{OA'}{OA}$ . This is called the **scale factor** of enlargement.

Name other ratios that are equal to the scale factor.

The scale factor of enlargement is defined as the ratio

$$\frac{\text{linear size of image}}{\text{corresponding linear size of object}}, \text{ or}$$

$$\frac{\text{Distance of an image point from the centre of enlargement}}{\text{Distance of a corresponding object point from the centre of enlargement}}$$

**Note:** This particular scale factor is often called the **linear scale factor** because it is the ratio of the lengths of two **line** segments.

## Construction of objects and images

### *Example 9.4*

*Draw any triangle XYZ. Taking a point O outside the triangle as the centre of enlargement, and with a scale factor 2, construct the image triangle X'Y'Z'.*

### *Solution*

*The following is the procedure of constructing the image:*

*(a) Draw triangle XYZ and choose a point O outside the triangle.*

*(b) Draw construction lines OX, OY and OZ, and produce them (Fig. 9.13*

*).*

*(c) Measure OX, OY and OZ. Calculate the corresponding lengths OX′, OY′ and OZ′; and mark off the image points.*
*From the definition of scale factor, it follows that* ***image distance = k × object distance*** *, where k is the scale factor. Thus, OX′ = 3OX, OY′ = 3OY and OZ′ = 3OZ.*

*(d) Join points X ′, Y ′, Z′ to obtain the image triangle.*

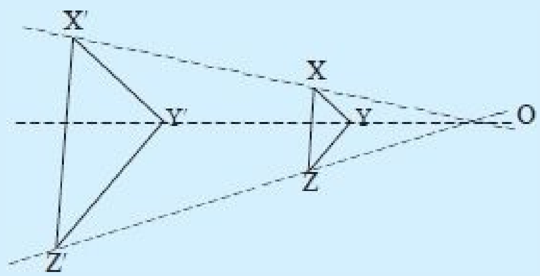



*Fig. 9.13*

We notice that

Under enlargement with a **scale factor greater than 1** ,

1. the object and the image are on the **same side** of the centre of enlargement,
2. the image is **larger** than the object, and
3. any line on the image is parallel to the corresponding line on the object.

## Exercise 9.3

1. Copy each of the shapes in Fig. 9.14 . Using the centre of enlargement indicated as X, enlarge each of the shapes with a scale factor of 3.

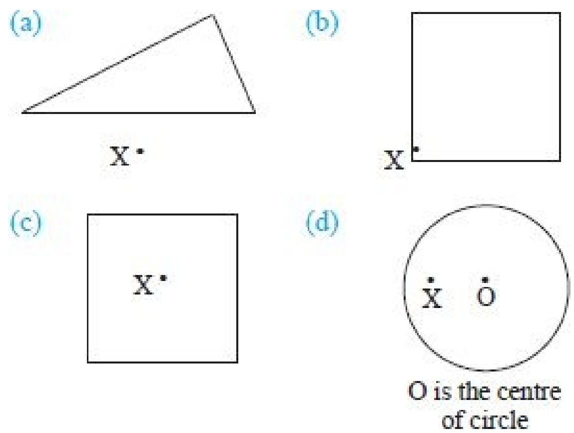



*Fig. 9.14*

2. In Fig. 9.15 , ΔOA′B′ is the enlargement of ΔOAB.
   (a) Find the scale factor of enlargement.
   (b) Calculate the lengths A′B′ and AA′.

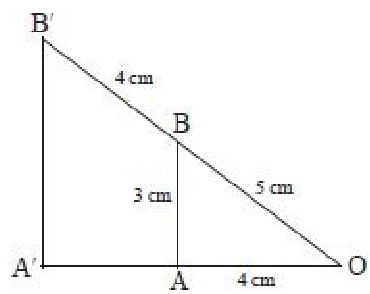



*Fig. 9.18*

## Fractional scale factor

If in Fig. 9.11 , we took (b) as the object, (a) would be the image under enlargement. In this case, the image is smaller than the object, so that the transformation is actually a **diminuation** . The scale factor here is a fraction since the size of the image is smaller than the size of the object.

***Example 9.5***

*Construct any triangle ABC. Choose a point O outside, and a bit far from the triangle. With O as the centre of enlargement, and with a scale factor* $\frac{1}{2}$ *, construct the image triangle* $A'B'C'$.

***Solution***

*The procedure is the same as that of Example 9.4 .*

*The only difference is that we multiply the object distance by $\frac{1}{2}$ (a fraction) so as to get the image distance, i.e.*

*Image distance = $\frac{1}{2}$ × Object distance.*

*The construction is shown in Fig. 9.16 .*

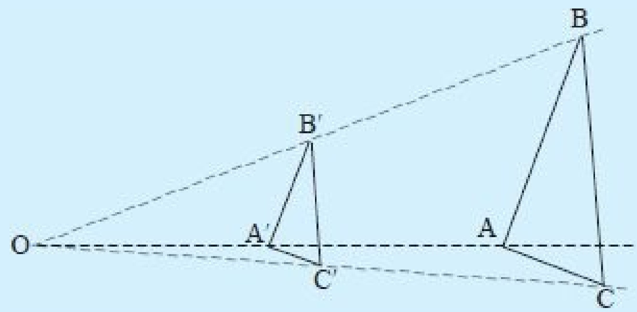



*Fig. 9.16*

We notice that:

Under enlargement with a **scale factor that is a proper fraction** ,

1. the object and the image are on the same side of the centre of enlargement.
2. the image is smaller than the object and lies between the object and the centre of enlargement.
3. any line on the image is parallel to the corresponding line on the object.

## Exercise 9.4

1. Copy Fig. 9.17 and locate the image of the flag under enlargement with centre O and scale factor (a) $\frac{1}{4}$ (b) 2.

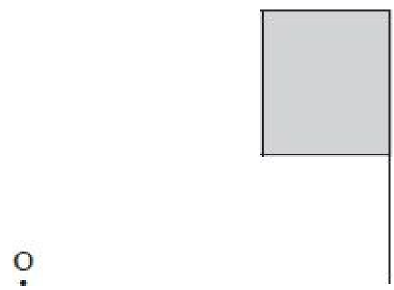



*Fig. 9.17*

2. In Fig. 9.18 , P′Q′ is an enlargement of PQ with centre O.

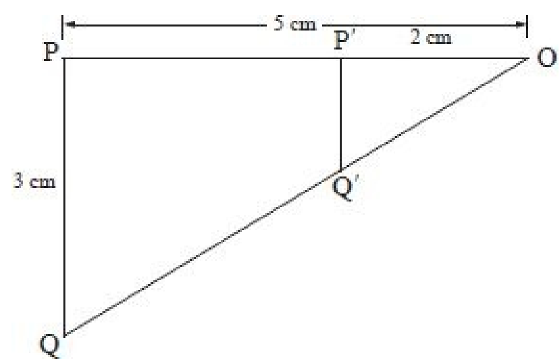



*Fig. 9.18*

(a) Find the scale factor of enlargement.
(b) Calculate the length of P′Q′.

3. Under enlargement of a polygon, what happens to the following if the scale factor is (i) 3, (ii) $\frac{1}{3}$ ?

(a) The lengths of corresponding sides
(b) Corresponding angles
(c) Shape
(d) Direction of lines

## Negative scale factor

Consider Fig. 9.19 , in which the centre of enlargement is O and both images of flag ABCD are 1.5 times as large as the object.

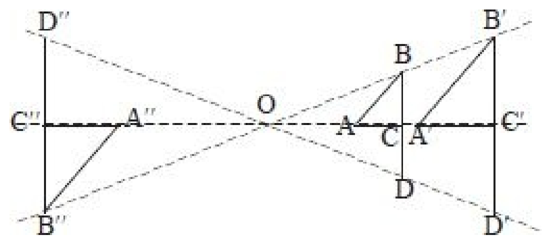



*Fig. 9.19*

A′B′C′D′ is the image of ABCD under enlargement with centre O and scale factor 1.5.

A″B″C″D″ is also an image of ABCD under enlargement with centre O. How is it different from A′B′C′D′?

OA = 1.2 cm and OA″ = 1.8 cm. OA and OA″ are on opposite sides of O. If we mark A″OA as a number line with O as zero and A as 1.2, then A″ is $^{-}1.8$ (Fig. 9.20 ).

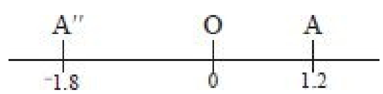



*Fig. 9.20*

We say that the scale factor is $^{-}1.5$ because

$^{-}1.5 \times 1.2 = {}^{-}1.8$.

We write OA″ = $^{-}1.5$ OA.

We notice that:

Under enlargement with a **negative scale factor** ,

1. the object and the image are on **opposite sides** of the centre of enlargement,
2. the object is larger or smaller than the object depending on whether the scale factor is greater than 1 and negative or a negative proper fraction,
3. any line on the image is **parallel** to the corresponding line on the object, but the image is **inverted** relative to the object.

## Enlargement in the Cartesian plane

### *Activity 9.1*

On squared paper, draw a quadrilateral with vertices A(0, 3), B(2, 3), C(3, 1) and D(3, $^{-}$2).

Copy and complete Table 9.1 for the coordinates of A′B′C′D′ and P′, the images of points A, B, C, D and a general point P(*a* , *b)* under enlargement with centre as the origin and the given scale factors.

| Scale factor | A′ | B′ | C′ | D′ | P′ |
|---|---|---|---|---|---|
| 2 | (0 , 6) | | | | $(2a , 2b)$ |
| $\frac{1}{2}$ | | | | (1.5 , $^{-}$1) | |
| $^{-}$2 | | | ($^{-}$6 , $^{-}$2) | | |
| $-\frac{1}{2}$ | | ($^{-}$1 , $^{-}$1.5) | | | |

*Table 9.1*

We notice that:

An enlargement with centre (0, 0) and scale factor *k* maps a point P(*a* , *b)* onto *P* ′*(ka , kb).*

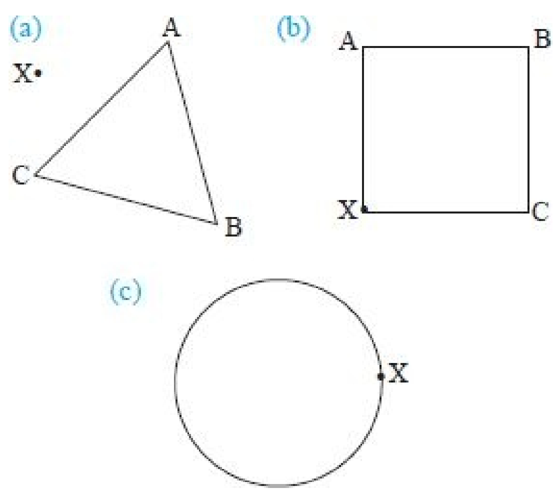



*Fig. 9.21*

**2.** In Fig. 9.22 , rectangle PQRS is an enlargement of rectangle ABCD with centre O.

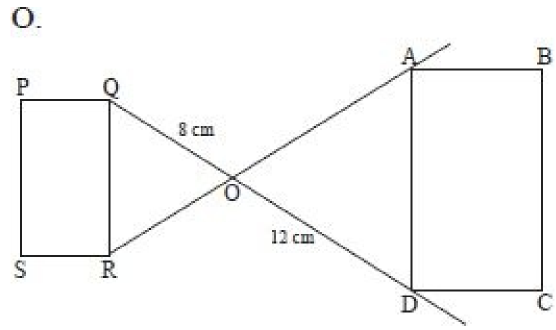



*Fig. 9.22*

(a) Find the scale factor of enlargement.
(b) Which point is the image of point
(i) A;
(ii) B;
(iii) C?
(c) Find the length of the diagonal of rectangle PQRS given that the length of the diagonal of rectangle ABCD is 15 cm.
(d) If rectangle ABCD is the image, what is the scale factor of enlargement?

**3.** Points A($^-2$, $^-1$), B(1, $^-1$), C(1, $^-4$) and D($^-2$, $^-4$) are vertices of a square. Without drawing, state the coordinates of the vertices of the image square under enlargement with centre origin and scale factor

(a) $^-4$
(b) $^-2$
(c) $^-1$
(d) $-\frac{1}{4}$

**4.** In a scale model of a building, a door which is actually 2 m high is represented as having a height of 2 cm.
(a) What is the scale of this model?
(b) Calculate the actual length of a wall which is represented as being 5.2 cm long.

**5.** When fully inflated, two balls have radii of 10.5 cm and 14 cm respectively. They are deflated such that their diameters decrease in the same ratio. Calculate the diameter of the smaller ball when the radius of the larger ball is 10 cm.

6. A tree is 6 m high. In photographing it, a camera forms an inverted image 1.5 cm high on the film. The film is then processed and printed to form a picture in which the tree is 6 cm high. Calculate the scale factors for the two separate stages.

## Proportional division of a line segment

A line segment can be divided into a given number of equal parts or in a given ratio as follows.

## Line divided into equal parts

### *Activity 9.3*

Divide line segment PQ in Fig. 9.23 (a) into 5 equal parts.


*Fig. 9.23*

*Procedure:*

(a) Through P, draw any line PK.

(b) Using a suitable radius on a pair of compasses, starting at P, mark off 5 equal lengths $PP_1$ , $P_1 P_2$ , $P_2 P_3$ , $P_3 P_4$ , $P_4 P_5$ , along PK.

(c) Join $QP_5$ .

(d) Using a ruler and set square, draw lines $P_1 Q_1$ , $P_2 Q_2$ , $P_3 Q_3$ , $P_4 Q_4$ parallel to $P_5$ Q. (Fig. 9.24 ).

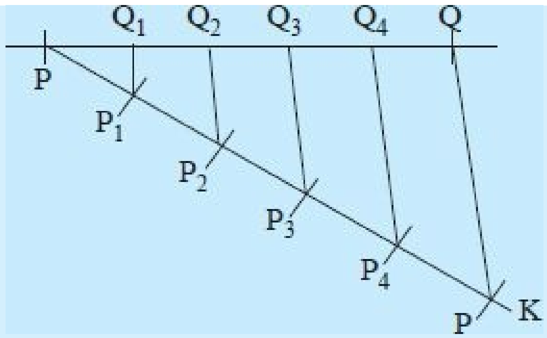



*Fig. 9.24*

$Q_1$ , $Q_2$ , $Q_3$ , $Q_4$ are the points that divide PQ into the required 5 equal parts.

## Line divided in a given ratio

### *Activity 9.4*

Draw line AB = 5 cm and divide it in the ratio 1 : 2.

*Procedure:*

(a) Draw line AB = 5 cm.

(b) Draw line AR (3 cm or 6 cm, or 9 cm, etc. long) at an acute angle to AB and divide it into 3 equal parts.

(c) Join RB.

(d) Lengths $AR_1$ , and $R_1$ R will then be in ratio 1 : 2. Hence draw $R_1$ X parallel to RB.

AX : XB = 1 : 2 (Fig. 9.25 )

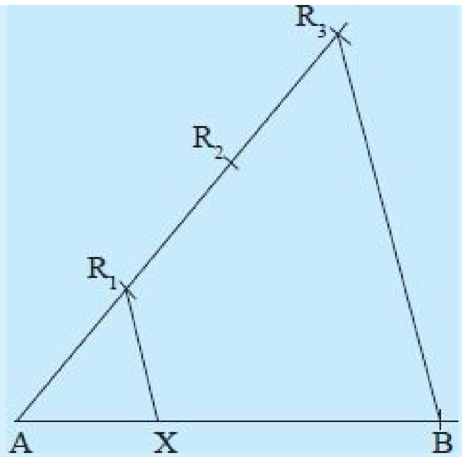



*Fig. 9.25*

## Exercise 9.6

1. Draw a line AB = 8 cm long. Find a point X on AB such that AX : XB = 5 : 4. Measure AX.
2. Draw a line PQ = 10 cm long. Find a point D on line PQ such that PD : DQ = 2 : 7. Measure DQ.
3. Using a ruler and a pair of compasses only, draw ΔABC such that AB = 8.4 cm, BC = 7.7 cm and AC = 10.5 cm. P and Q are points on AB and AC respectively such that AP : PB = AQ : QC = 3 : 4. Measure PQ and find the ratio
   (a) PQ : BC
   (b) AP : AB.
4. Construct ΔABC such that AB = 8 cm, BC = 10 cm and CA = 9 cm. Find a point X on AB such that AX = XB. Through X draw line XY parallel to BC and meeting AC at Y. Measure XY. Find the ratio
   (a) XY : BC
   (b) AX : AB

# 10 INEQUALITIES I

## Inequalities

We are familiar with the equals sign, =. Remember that an algebraic statement which has an equals sign, e.g. $3a = 6$, is called an **equation** .

In this section, we are concerned with statements which involve the following symbols:

> meaning 'greater than', e.g. $6 > 2$ means 6 is greater than 2.

< meaning 'less than', e.g. $3 < 7$ means 3 is less than 7.

≥ meaning 'greater than or equal to' (or 'not less than').

≤ meaning 'less than or equal to', (or 'not greater than').

Statements containing these symbols are called **inequalities** or **inequations** .

For example, $x < 2$, $x \geq 3$, $y \leq {}^-3$, $y > 10$ are inequalities. These are also called **simple inequality statements** .

A statement such as $x > 2$ means 'all numbers that are greater than 2', which is a range of values. Just as we represent individual numbers on a number line, we can also represent such a range of numbers on a number line as shown in the following examples.

***Example 10.1***

*Illustrate each of the following on a number line:*

(a) $x > 3$

(b) $x \geq 3$

(c) $x < {}^-2$

(d) $x \leq {}^-2$

***Solution***

*(a) $x > 3$:*

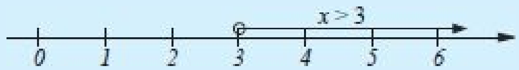



*Fig. 10.1*

*In Fig. 10.1 , the number 3 is not included in the list of numbers to the right of 3. The heavy arrow shows that the values of x go on without end. The open dot ° is used to indicate that 3 is not included.*

*(b) $x \geq 3$:*

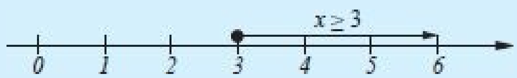



*Fig. 10.2*

*In Fig. 10.2 , the number 3 is included in the list of the required numbers. The closed dot • is used to show that 3 is part of the list.*

*(c) $x < {}^{-}2$:*

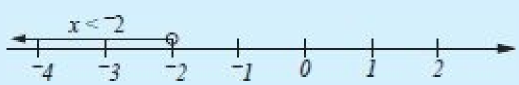



*Fig. 10.3*

*In Fig. 10.3 , the number $^{-}2$ is not included.*

*(d) $x \leq {}^{-}2$:*

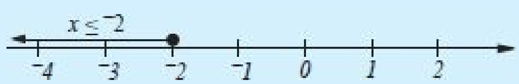



*Fig. 10.4*

*In Fig. 10.4 , the number $^{-}2$ is included.*

## Exercise 10.1

1. Rewrite each of the following statements using either <, ≤, > or ≥ instead of the words.

   (a) 5 is less than 7
   (b) 5 is greater than 2
   (c) $^{-}1$ is less than 0
   (d) $^{-}2$ is greater than $^{-}3$
   (e) $x$ is greater than or equal to 4
   (f) $y$ is less than or equal to $^{-}5$
   (g) $a$ is not less than 3
   (h) $b$ is not greater than 0
   (i) $^{-}1$ is not less than $p$
   (j) 10 is not greater than $q$

2. Copy and complete each of the following statements by inserting the correct symbol, < or >.

   (a) $7 + 3 \square 11$
   (b) $6 - 3 \square 5$
   (c) $13 + 2 \square 16$
   (d) $12 - 4 \square 3$
   (e) $3 \times 4 \square 8$
   (f) $^{-}4 \times 3 \square 12$
   (g) $^{-}6 \times {}^{-}2 \square {}^{-}6 \times 2$
   (h) $39 \div 3 \square 39 \div {}^{-}3$

3. Illustrate each of the following on a number line.

   (a) $x > {}^{-}5$
   (b) $x < 0$
   (c) $x \geq {}^{-}3$
   (d) $x \leq 4$
   (e) $x > {}^{-}0.5$
   (f) $x \leq 2.5$

## Compound statements

Sometimes, two simple inequalities may be combined into one **compound statement** such as $a < x < b$ . This statement means that $a < x$ and $x < b$ or $x > a$ and $x < b$ .

### *Example 10.2*

*Write the following pairs of simple inequality statements as compound statements and illustrate them on number lines.*

*(a)* $x \leq 3, x > {}^-3$

*(b)* $x > {}^-1, x < 2$

### *Solution*

*(a)* $x \leq 3, x > {}^-3$ *becomes* ${}^-3 < x \leq 3$ *(Fig. 10.5 ).*

$\therefore$ *x lies between* ${}^-3$ *and 3, and 3 is included.*

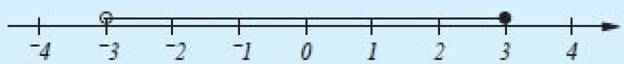



*Fig. 10.5*

*(b)* $x > {}^-1, x < 2$ *becomes* ${}^-1 < x < 2$ *(Fig. 10.6 ).*

$\therefore$ *x lies between* ${}^-1$ *and 2.*

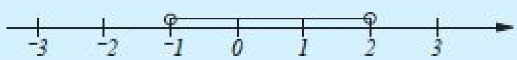



*Fig. 10.6*

## Exercise 10.2

Write each of the following pairs of simple statements as a compound statement and illustrate the answer on a number line.

1. (a) $x > 2, x < 4$
   (b) $x < 3, x \geq 0$
   (c) $x \leq 5, x > {}^-2$
   (d) $x \geq {}^-1, x \leq 1$
   (e) $x < 1.5, x \geq {}^-0.5$
   (f) $x \leq 2.2, x \geq 1.8$
2. (a) $x < \frac{1}{4}, x \geq 0$
   (b) $x \geq \frac{3}{4}, x < 2\frac{1}{4}$

(c) $x < 3\frac{1}{2}$ , $x > 2\frac{1}{2}$
(d) $x > \frac{1}{5}$, $x < \frac{2}{3}$
(e) $x \geq {}^{-}0.75$, $x \leq 0.75$
(f) $x < 4\,\frac{1}{2}$ , $x > -\frac{1}{2}$

## Forming inequalities from word statements

Inequality symbols can be used to change word statements into algebraic statements.

### *Example 10.3*

*Mary has 25 books. Jane has more books than Mary. Write an algebraic statements for this.*

### *Solution*

*If Jane has b books then b > 25.*

### *Example 10.4*

*The distance from Blantyre to Salima is 260 km and that from Blantyre to Zomba is 80 km. If Malindi to Blantyre is a shorter distance than Blantyre to Salima, and Malindi to Blantyre is a longer distance than Blantyre to Zomba, write an algebraic statement for this.*

### *Solution*

*Let the distance from Blantyre to Malindi be x km.*

*Then, x < 260 and x > 80*

*Hence, 80 < x < 260*

### *Example 10.5*

*The area of a square is greater than 36 cm$^2$ . Write an inequality for (a) the length (b) the perimeter of the square.*

### *Solution*

*We must first define our variables, just as we do when forming equations.*

*Let the length of the square be x cm.*

*Area of the square* $= x^2$ .

(a) $x^2 > 36$

$\Rightarrow \sqrt{x^2} > \sqrt{36}$

$\Rightarrow x > 6$

(b) *Perimeter* $= 4x$

*Since* $x > 6$,

*then* $4 \times x > 4 \times 6$

*i.e.* $4x > 24$

## Exercise 10.3

Write down the statements as mathematical sentences using appropriate symbols.

1. When two is added to a certain number, the result is greater than ten.
2. When five is added to a number, the result is less than twice the number.
3. Multiplying a number by six, then adding five, gives a greater result than multiplying the number by five, then adding six.
4. The sum of three consecutive whole numbers is more than 260.
5. The area of a square is less than its perimeter.
6. The square of a number is less than the number cubed.
7. The radius of a circle is not more than 4 cm. What can you say about the circumference?

## Solution of linear inequalities in one unknown

Solving an inequality means obtaining all the possible values of the unknown which make the statement true. This is done in much the same way as solving an equation.

***Example 10.6***

*Solve the following inequalities*

(a) $x - 3 < 7$

*(b) $x + 5 > 11$*

## *Solution*

*(a) $x - 3 < 7$*

$\Rightarrow x - 3 + 3 < 7 + 3$ *(Adding 3 to both sides)*

$\Rightarrow x < 10$

*Thus, $x < 10$ is the solution of the inequality*

*$x - 3 < 7$.*

*(b) $x + 5 > 11$*

$\Rightarrow x + 5 - 5 > 11 - 5$ *(subtracting 5 from both sides)*

$\Rightarrow x > 6$.

Adding (or subtracting) the same number to (or from) both sides of an inequality does not change it.

## *Example 10.7*

*Solve the following inequalities.*

*(a) $3x - 4 \geq 5$*

*(b) $\frac{1}{4}$ x + 5 $\leq$ 14*

## *Solution*

*(a) $3x - 4 \geq 5$*

$\Rightarrow 3x - 4 + 4 \geq 5 + 4$

$\Rightarrow 3x \geq 9$

$\Rightarrow \frac{3x}{3} \geq \frac{9}{3}$ *(Dividing both sides by 3)*

$\Rightarrow x \geq 3$

*(b) $\frac{1}{4} + 5 \leq 14$*

$\Rightarrow \frac{1}{4} x + 5 - 5 \leq 14 \Rightarrow \frac{1}{4} x \leq 9$

$-5$

$\Rightarrow \frac{1}{4} x \times 4 \leq 9 \times 4 \Rightarrow x \leq 36$

Multiplying or dividing both sides of an inequality by the same positive number does not change it.

## Multiplication and division of inequalities by negative numbers

We know that $8 < 10$.

Consider multiplying both sides of this inequality by any negative number, say $^{-}4$.

LHS = $8 \times {}^{-}4 = {}^{-}32$

RHS = $10 \times {}^{-}4 = {}^{-}40$

We know that $^{-}32$ is greater than $^{-}40$, i.e. $8 \times {}^{-}4 > 10 \times {}^{-}4$

Thus, the **inequality is reversed** .

Similarly, $8 \div {}^{-}4 = {}^{-}2$, and

$10 \div {}^{-}4 = {}^{-}2.5$

We know that $^{-}2 > {}^{-}2.5$

i.e. $\frac{8}{^{-}4} > \frac{10}{^{-}4}$

Thus, the **inequality is reversed** .

In general, multiplying or dividing both sides of an inequality by a negative number reverses the inequality sign.

### *Example 10.8*

*Solve the inequality $3 - 2x \geq 15$.*

### *Solution*

$3 - 2x \geq 15$

$\Rightarrow 3 - 2x - 3 \geq 15 - 3$

$\Rightarrow {}^{-}2x \geq 12$

$\Rightarrow \frac{{}^{-}2x}{{}^{-}2} \leq \frac{12}{{}^{-}2}$

$\Rightarrow x \leq {}^{-}6$

## Exercise 10.4

Solve the following inequalities and represent the solutions on number lines.

1. (a) $x + 4 > 11$
   (b) $x - 6 \leq 5$
2. (a) $2x - 8 \leq 4$
   (b) $3x + 4 > 19$
3. (a) $3 > 4x - 2$
   (b) $7 \leq 5x + 12$
4. (a) $3 - 2x < 5$
   (b) $4 - 5x \geq {}^{-}11$
5. (a) $\frac{1}{3}x - 3 > 4$
   (b) $\frac{1}{5}x + 2 < 1$
6. (a) ${}^{-}4 > 2 - \frac{1}{7}x$
   (b) $-\frac{2}{3}x + 4 \leq {}^{-}6$
7. (a) $4m - 3 < 7m$
   (b) $2m + 1 \geq 5m - 10$
8. (a) $2 - 2p > 13 - 3p$
   (b) $\frac{1}{2} + p < 4p + \frac{1}{4}$
9. (a) $\frac{1}{3}q > 2 - 4q$
   (b) $\frac{1}{4}q + 2 < 8 - \frac{2}{3}q$
10. (a) $-\frac{1}{9}r < 5$
    (b) $4 - \frac{3}{4}r \geq 3 - \frac{1}{3}r$
11. (a) $2(1 + x) + 3(x - 2) \geq 25$
    (b) $3(4 - 3x) - (5x - 3) \leq 2$

## Solving simultaneous inequalities

Inequalities that must be satisfied **at the same time** are called simultaneous inequalities.

### *Example 10.9*

*Solve the following pair of simultaneous inequalities.*

$3 - x < 5,\ 2x - 5 < 7$

### *Solution*

$3 - x < 5$

$\Rightarrow {}^{-}x < 2$

$\Rightarrow x > {}^{-}2$ ………..(i)

*Also* $2x - 5 < 7$

$\Rightarrow 2x < 12$

$\Rightarrow x < 6$ ………..(ii)

*Combining (i) and (ii), we have* ${}^{-}2 < x < 6$.

*Thus x lies between* ${}^{-}2$ *and 6.*

*This is represented on a number line as in Fig. 10.7 .*

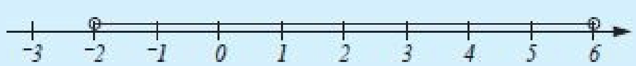



*Fig. 10.7*

### *Example 10.10*

*Solve the inequality* $3x - 2 < 10 + x < 2 + 5x$

### *Solution*

$3x - 2 < 10 + x < 2 + 5x$

*Split the inequality into two simultaneous inequalities as:*

$3x - 2 < 10 + x$ ………..(i)

*and* $10 + x < 2 + 5x$ ………..(ii)

*Solve each separately.*

$3x - 2 < 10 + x$

$\Rightarrow 3x - x < 10 + 2$

$\Rightarrow 2x < 12$

$\Rightarrow x < 6$ ……………(iii)

$10 + x < 2 + 5x$

$\Rightarrow 10 - 2 < 5x - x$

$\Rightarrow 8 < 4x$

$\Rightarrow 2 < x$ ……………(iv)

*Combine (iii) and (iv) to get* $2 < x < 6$

*This is represented on a number line as in Fig. 10.8 .*

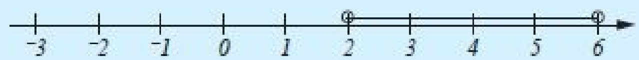



*Fig. 10.8*

## Exercise 10.5

Solve the following simultaneous inequalities and represent each solution on a number line.

1. (a) $2x < 10,\ 5x \geq 15$

   (b) $3x \leq 9,\ 2x > 0$

2. (a) $x + 7 < 0,\ x - 2 > {}^{-}10$

   (b) $x \geq 3,\ 2x - 1 \leq 13$

3. (a) $4x - 33 < -1,\ {}^{-}2 < 3x + 1$

   (b) $2x - 5 < 22 \leq 5x - 6$

4. (a) $3x - 4 < 8 + x < 2 + 7x$

   (b) $6x + 2 < 3x + 8 < 27x - 1$

# 11 TRAVEL GRAPHS I

In Form 1, we learnt how to draw and interpret linear graphs. In this chapter, we will learn how to draw and interpret travel graphs. **Travel graphs** are graphical representations of the motion of an object from one point to another such as the distance – time graphs.

## Drawing graphs

The following are key points necessary for drawing graphs that are easy to read and interpret.

(a) Choose as large a scale as the paper allows. This makes plotting and reading easy. Ensure that you accommodate all the data in the table.

(b) The quantity whose values are selected (independent variable) should be placed along the horizontal axis, while the quantity whose values are observed or calculated (dependent variable) should be placed along the vertical axis.

(c) Graduate and clearly label the axes and write the units used.

(d) Write a brief explanatory heading (title) above the graph.

(e) If two graphs are drawn on the same axes, label each clearly.

In order to draw the graph of a given relation, we need to draw a table of values giving values of the independent variable and corresponding values of the dependant variable.

Consider the following example.

***Example 11.1***

*Form a table of values for the relation $y = x + 4$ and draw its graph.*

***Solution***

*$y = x + 4$*

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $y$ | 4 | 5 | 6 |

*Table 11.1*

*Note that in Table 11.1 , only three values of x are chosen from which corresponding values of y are calculated. This is because the equation is linear, and so two points are sufficient for drawing the graph: The third point is only for confirmation that we have the correct line.*

*Fig. 11.1 is the required graph.*

*A graph of $y = x + 4$*

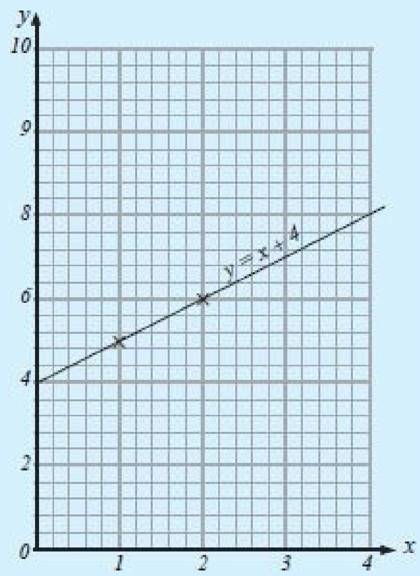



*Fig. 11.1*

**Note** that when dealing with graphs of motion (travel graphs), tables of values are often provided.

## Distance

The length from one point to another is known as the **distance** . It is measured in **metres (m)** . For long distances, kilometre (km) is used. For example, the distance between Lilongwe and Blantyre is 311 km, while the shortest distance in an athletics race is 100 m. Other shorter distances may be measured in centimetres (cm) or in millimetres (mm).

## Speed

When an object moves a certain distance, the distance it moves divided by the time taken gives the **speed** , i.e.

$$\text{Speed} = \frac{\text{Distance covered}}{\text{Time taken}}$$

If the distance is in kilometres and the time is in hours, then the speed is given in **kilometres per hour** (**km/h** or **kmh** $^{-1}$ or **kph** ).
If the distance is in metres and the time in seconds, then speed is given in **metres per second** (**m/s or ms** $^{-1}$ ).
Speed is therefore the rate of change of distance per unit time.

### *Example 11.2*

*A man walked 10.8 km at 5.4 km/h. Find the time taken.*

### *Solution*

$$\begin{aligned} \textit{Speed} &= \frac{\textit{Distance covered}}{\textit{Time taken}} \\ \therefore \textit{Time taken} &= \frac{\textit{Distance covered}}{\textit{Speed}} \\ &= \frac{10.8 \textit{ km}}{5.4 \textit{ km/h}} = 2 \textit{ h} \end{aligned}$$

### *Example 11.3*

*Chimodzi ran in 12.5 s at a speed of 8 m/s.*
*What distance did he cover?*

### *Solution*

$$Speed = \frac{Distance}{Time}$$

$$\begin{aligned} Distance &= Speed \times Time \\ &= 8\ m/s \times 12.5\ s \\ &= 100\ m \end{aligned}$$

## *Example 11.4*

*A train leaves town A and travels towards town B at 48 km/h. At the same time, another train leaves town B and travels towards A at a speed of 52 km/h. If the two towns are 500 km apart, find*

*(a) how far apart the trains are after travelling for 45 minutes.*

*(b) how far, from town A, they will be when they meet.*

*(c) how long they will take to meet.*

## *Solution*

(a) $45\ min = \frac{3}{4}\ h.$

$$Since\ Speed = \frac{Distance}{Time}$$

$$Distance = Time \times Speed$$

*Distance = Time × Speed*

*Train from town A will have travelled a distance*

$$= \frac{3}{4} \times 48$$

$$= 36\ km$$

*and train from town B will have travelled a distance*

$$= \frac{3}{4} \times 52 = 39\ km$$

*∴ the distance apart = 500 – (36 + 39) km*

*= 425 km*

*(b) The problem may be sketched as in Fig 11.2 , where C is the meeting point and x is the distance travelled by train from town A by the time they meet.*

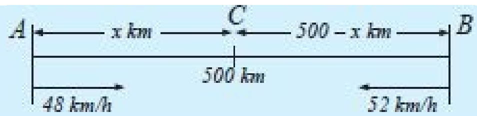



*Fig. 11.2*

$$\text{Since Speed} = \frac{\text{Distance}}{\text{Time}}, \quad \text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

*The trains start at the same time. So they have travelled the same length of time by the time they meet.*

$$\therefore \quad \text{time taken} = \frac{x}{48} = \frac{500 - x}{52}$$

$$\therefore \quad 48(500 - x) = 52x$$

$$\Rightarrow 24\,000 - 48x = 52x$$

$$\Rightarrow 24\,000 = 52x + 48x$$

$$\Rightarrow 24\,000 = 100x$$

$$\therefore x = 240 \text{ km}$$

*i.e. the trains meet 240 km away from town A.*

(c) $\text{Time taken to meet} = \frac{x}{48}\ h$

$$= \frac{240}{48}\ h$$

$$= 5\ h$$

## Exercise 11.1

1. A vehicle crosses a bridge $\frac{1}{10}$ km long at a speed of 72 km/h. Calculate the time taken in seconds.
2. A train takes 14 seconds to pass a signal post at a speed of 13 m/s. Calculate the length of the train.
3. A taxi travelling at 60 km/h took 2 h 25 min between two towns. Find the distance between the towns.
4. Mr. Banda took a total of 2 h to walk from his home to his workplace and back. If his speeds for the to-and-fro journeys were 2 km/h and 3 km/h respectively, find the distance between his home and workplace.
5. Ng'ambi travelled between two towns at a speed of 72 km/h. Atusaye took half an hour less to travel the same distance at a speed of 90 km/h. Find the distance between the two towns.

6. One train leaves a station 2 h before a second train which follows in the same direction. If the first train travels at 40 km/h and the second at 48 km/h, how long will it take for the second train to catch up with the first?
7. Kevin left town A at 5.30 a.m. and travelled towards town B at a speed of 48 km/h.
   At the same time, Kezia left town B and travelled towards town A at a speed of 64 km/h. If the distance between the two towns is 588 km,
   (a) at what time did they meet?
   (b) how far from town B was their meeting point?
   (c) how far apart were they after travelling for 2 h?
8. How long will a train 70 m long travelling at 45 km/h take to pass a stationary train 80 m long?
9. A man had walked two-thirds way across a bridge when he sighted an approaching train 60 m away. He ran back only to reach the end of the bridge at the same time as the train. If the train was moving at 25 m/s and the man ran at 10 m/s, find the length of the bridge.

## Distance-time graphs

Two cars X and Y started moving from a particular point. The distances covered and the corresponding times taken are recorded in Table 11.2 (a) and (b) .

Car X

(a)

| Time $t$ (min) | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Distance $s$ (km) | 0 | 1 | 2 | 3 |

Car Y

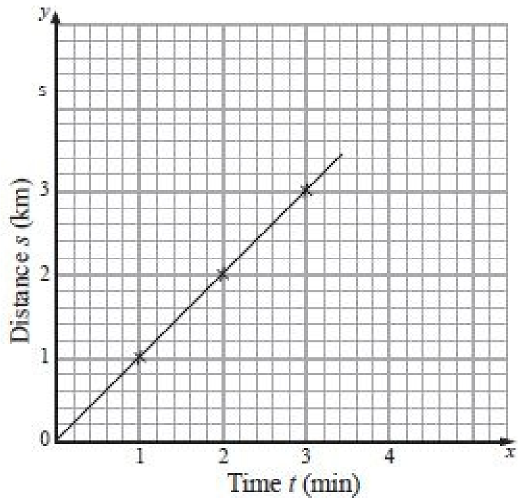
(b)

| Time $t$ (min) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| Distance $s$ (km) | 0 | 0.5 | 1.8 | 3 | 3 |

*Table 11.2*

Fig. 11.3 shows the graphs of the relationships between the distance (s) and time (t) for the two cars.

(a) Distance – time graph for Car X




(b) Distance - time graph for Car Y

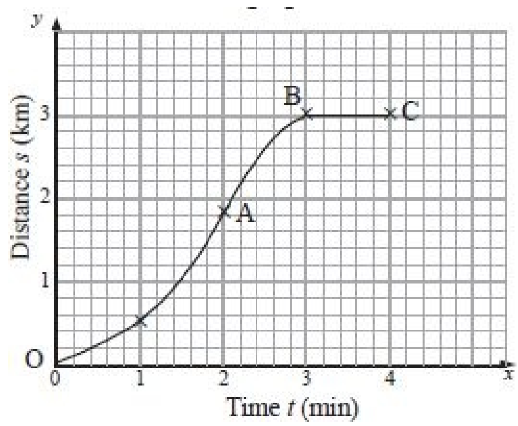



*Fig. 11.3 Distance-time graphs*

Remember, **gradient** is the steepness of a line. Hence, the steeper the line, the higher the gradient.

For distance – time graphs, gradient is equivalent to the speed of the moving object.

The graph for Car X is a straight line since the speed of Car X is constant.

Thus, the graph of motion under constant speed is a straight line.

How would the graph of motion look like for a stationary object?

The graph of Car Y (Fig. 11.3(b) ) has three sections:

- From O to A, the gradient is increasing. This shows that the speed was increasing for the first 2 minutes.
- From A to B the curve is becoming less steep i.e. the gradient is decreasing. This shows that between the times $t = 2$ and $t = 3$, the speed was decreasing.

- From B to C, the gradient is zero. This shows that the speed is zero, thus the car had stopped 3 km from the starting point.

Thus, the graph of motion for a body with varying speed is a curve.

***Example 11.5***

*Table 11.3 shows the distance covered and the corresponding time taken for a cyclist moving downhill.*

| *Time t (s)* | *0* | *0.5* | *1* | *1.5* | *2* | *2.5* | *3* | *3.5* | *4* |
|---|---|---|---|---|---|---|---|---|---|
| *Distance s (m)* | *0* | *2* | *8* | *17* | *28* | *41* | *60* | *81* | *104* |

*Table 11.3*

*(a) Draw a distance-time graph.*

*(b) Find the distance travelled in the first 3.2 seconds.*

***Solution***

*(a) Fig. 11.4 shows the graph.*

*A distance – time graph for a cyclist moving downhill*

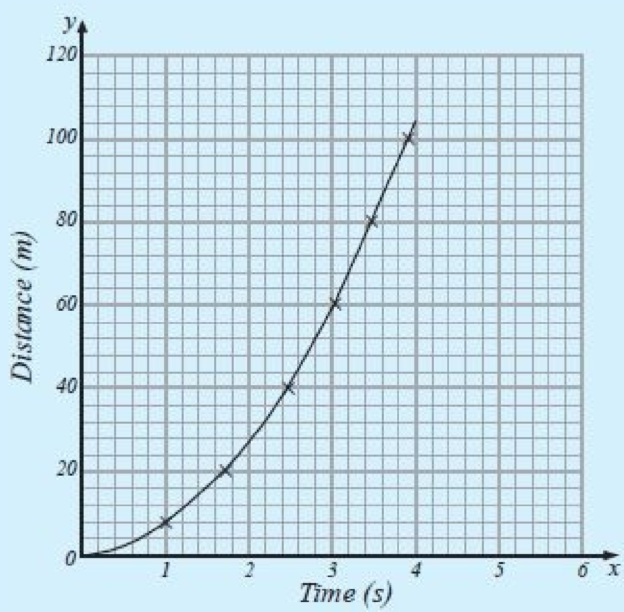



*Fig. 11.4*

*(b) When t = 3.2 s, distance = 68 m*

When drawing a distance-time graph, always ensure that time is plotted on the horizontal axis.

## Exercise 11.2

1. A man travelled part of the journey to his workplace by bus, then got a lift on a motor bike before walking the rest of the journey (see Fig. 11.5 ).

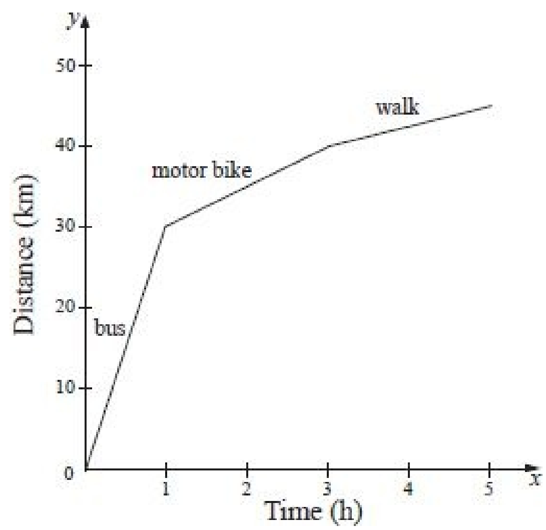



*Fig. 11.5*

   What is the distance between his home and his workplace and how long did he take to travel the whole distance?

2. Agnes travelled a distance of 32 km by car at a speed of 80 km/h. She then cycled a distance of 7.4 km at 10 km/h and rested for 10 minutes before taking the rest of the journey on foot, a distance of 2 km in half an hour. Draw a distance-time graph for the entire journey.

3. A marble was rolled down a sloping plank and the values in Table 11.4 were recorded.

| Time $t$ (s) | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|
| Distance $s$ (m) | 1.58 | 2.24 | 2.74 | 3.16 | 3.54 | 3.87 |

*Table 11.4*

Draw the graph of $s$ against $t$.

4. A cyclist rides at a constant speed of 10 m/s from his home to a market a distance of 1 km away and then immediately returns home at the same constant speed. Draw the distance-time graph for this motion.

5. In a physics experiment, a ball is rolled down a slope. The values in Table 11.5 are recorded.

| Time $t$ (s) | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| Distance $s$ (m) | 0 | 0.7 | 2.8 | 6.3 | 11.2 | 17.2 | 25.2 |

*Table 11.5*

Draw a graph of $s$ against $t$ .

6. A bus leaves town X at 1200 h for town Y at a speed of 64 km/h. At the same time, another bus leaves town Y for town X travelling at a constant speed of 60 km/h. Given that the distance between the two towns is 186 km, draw a distance - time graph showing both journeys and from your graph find when and where the buses pass each other.

7. A coach leaves town P and travels to town B a distance of 75 km at a speed of 60 km/h.

   After a brief stop over at B for 30 minutes, it continues to town D 35 km away at a speed of 35 km/h. At 2100 h, another coach leaves town D for town P and travels without stopping at a speed of 55 km/h.

   (a) Draw a distance-time graph to show the two journeys.

   (b) From your graph find the distance from D where the coaches pass each other.

8. During an athletics trial for Olympic games, Hope recorded the following results in Table 11.6 .

| Distance *(m)* | 150 | 200 | 440 | 600 | 880 | 1 000 |
|---|---|---|---|---|---|---|
| Time *(s)* | 14.6 | 19.4 | 47.0 | 70.8 | 112.2 | 132.4 |

*Table 11.6*

Use a distance-time graph to answer the following questions.

   (a) Find the possible record for

(i) 500 m
(ii) 750 m

(b) How far did the athlete run in

(i) 1 min?
(ii) 2 min?

9. Fig. 11.6 shows a travel graph of a cart.
By observation, how can you tell the part of the journey which corresponds to the highest speed.

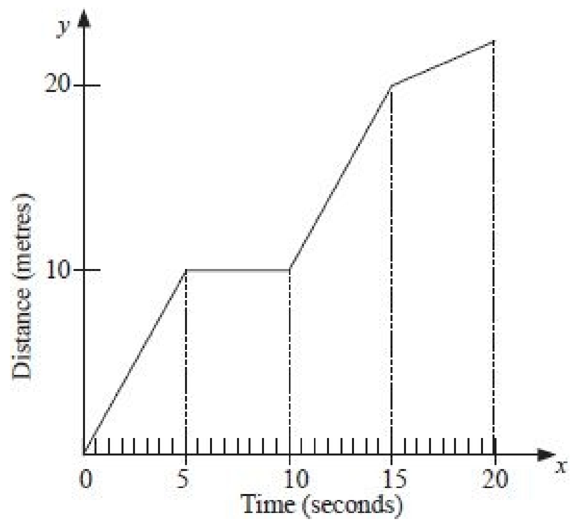



*Fig. 11.6*

## Interpretation of travel graphs

### Distance-time graph

Fig. 11.7 is a distance-time graph. It shows Onani’s distance from his house, where he started his journey, over a period of time. What can we deduce about his journey from the graph?

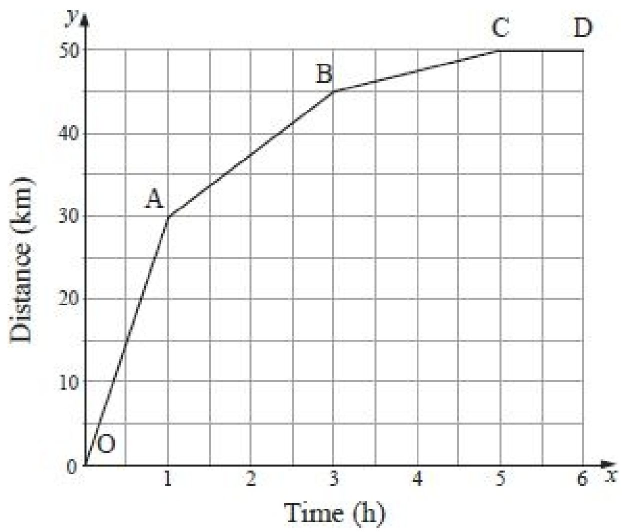



*Fig. 11.7*

The following is the interpretation from the graph:

Point O is the starting point, i.e. Onani's house.

OA — Onani travels at a constant speed so that after 1 h, he is 30 km from his house.

AB — He travels at a constant speed and takes 2 h to travel (45 – 30) km = 15 km. His speed is lower than in the first stage OA.

At B, he is 45 km from his house, having taken a total of 3 h.

BC — He takes 2 h and travels 5 km in this time. His speed is much slower than before. The speed is constant.

At C, he is 50 km away from his home.

CD — The distance is not changing. This means that he is not moving during that hour.

## Exercise 11.3

Describe the motions in the travel graphs in Fig. 11.8 .

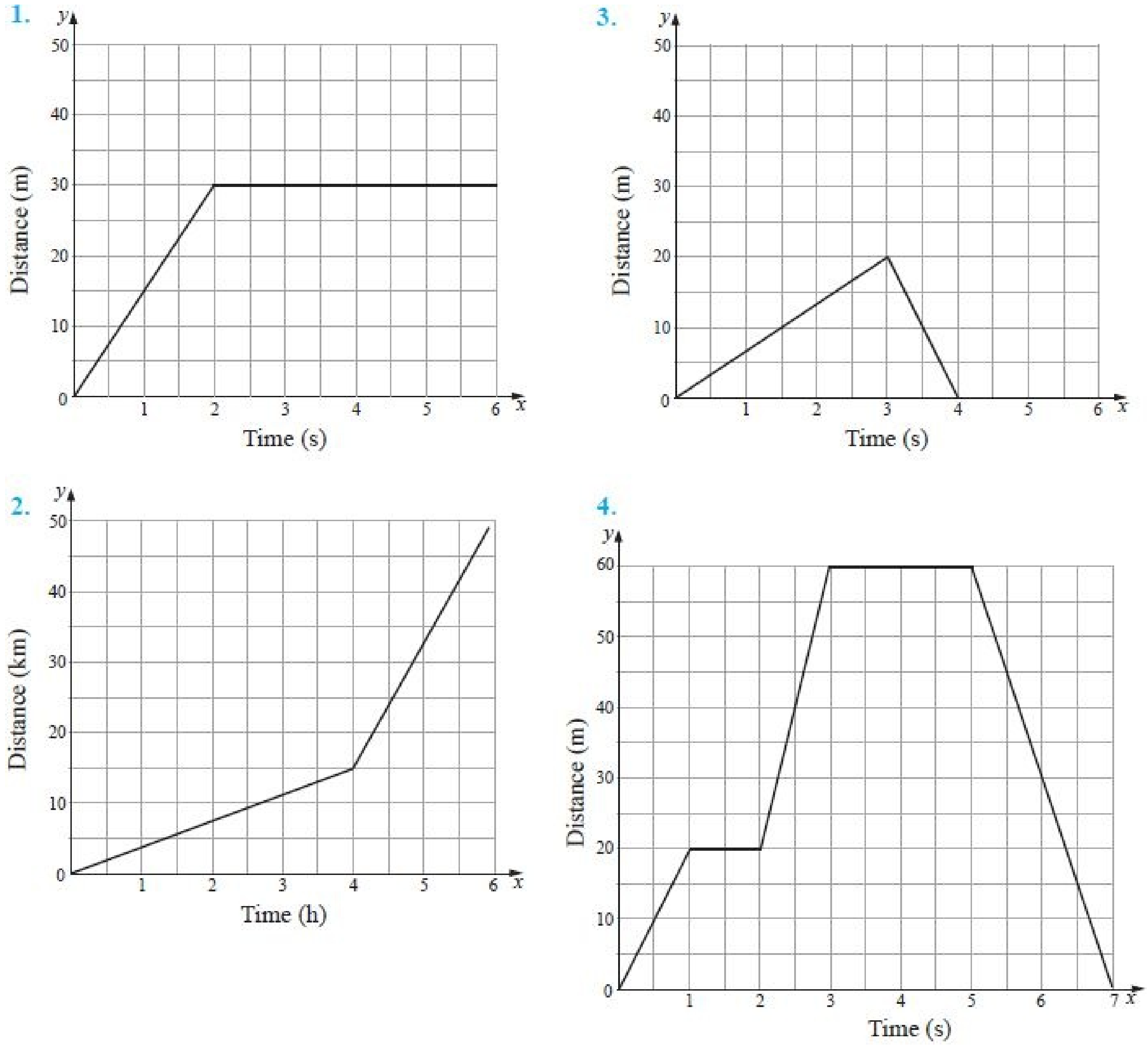

1.
Distance (m)
Time (s)
2.
Distance (km)
Time (h)
3.
Distance (m)
Time (s)
4.
Distance (m)
Time (s)

*Fig. 11.8*

# 12 PROBABILITY I

## What is probability?

Consider the following statements:

1. The sun will rise tomorrow morning.
2. If you put a shirt in water, it will get wet.
3. If you put your exercise book in fire, it will burn.

Under normal circumstances would you imagine the sun not rising in the morning or putting a shirt in water and it remains dry, or putting an exercise book in the fire and it does not burn? Since these events will definitely take place, they are said to be **sure** or **certain** events.

State any three other events that are certain.

Now think about the following statements:

1. It will rain tomorrow.
2. The national football team will win their next match.
3. Mala will win the elections.

These statements are about events which may or may not take place. We cannot be very sure or certain that they will or will not take place. Such events are said to be **uncertain events** . We use words like **probable** , **likely** , **unlikely** , **chance** etc, when talking about such events. For example, we could say, “It is **likely** to rain tomorrow.” or “It will **probably** rain tomorrow.” or “There is a **chance** of it raining tomorrow.”

What can you say about the following?

1. In the last two weeks it has been raining everyday. It will rain tomorrow.
2. The national football team has won 5 out their last 6 matches. The team will win the next match.
3. This medicine has helped 8 out of 10 patients who had a problem similar to yours. Try it, you will be helped.

Note that in these statements, there are numerical values that help us to predict whether the events are more or less likely to occur.

For example, from the fact that the national team has won 5 out of 6 matches, we could predict that their chance of winning the next one is very high. If 8 out of 10 patients have been helped by a certain medicine we are able to predict that another such patient could be helped by the same medicine.

We usually assign numerical values to our predictions. Thus in the case of the national football team, we would say that the chance of the team winning is $\frac{5}{6}$ . Similarly, in the case of the medicine, we would say that the chance of it helping is $\frac{8}{10}$ .

The numerical values $\frac{5}{6}$ and $\frac{8}{10}$ are called **probabilities** .

Thus:

**Probability** is the branch of mathematics in which appropriate numerical values are assigned as measures of the chances of **uncertain events** occuring or not occuring. The numerical values assigned are called **probabilities** (singular: probability), derived from the word “probable”.

Thus we can say: “The probability of the national football team winning is $\frac{5}{6}$ .”
“The probability of the medicine helping a patient is $\frac{8}{10}$ .”

The knowledge of probability has many practical uses. The following are a few examples.

1. An agricultural officer may need to know whether a difference in crop yield is due to different conditions or just to chance.
2. A manufacturer may need to know what percentage of his/her products needs to be tested to check whether the machines are still working efficiently.
3. A medical researcher may need to know what percentage of tests are needed to decide that a newly discovered drug produces the desired effect.

State more such examples.

## Experimental probability

For us to be able to define experimental probability, let us consider the Malawian coin. The side with the Coat of Arms will be referred to as **heads** (H) and the other side as **tails** (T). The tossing of a coin and recording the observation as either H or T is referred to as an **experiment** . Each toss of a coin is called a **trial** . Each observation or result of a trial is called an **outcome** . Thus, H is one outcome and T is another outcome. A **fair coin** is one which does not favour any side when tossed.

Suppose a fair coin is tossed 100 times and the number of heads and tails showing are recorded as 48 and 52 respectively.

The number of trials is 100.

The number of times "heads" appears (outcomes) is 48 and the number of times "tails" appears (outcomes) is 52.

The fraction $\frac{48}{100}$ is referred to as the **experimental probability** of obtaining heads.

In short, we write P(Heads) or P(H) = $\frac{48}{100}$ .

Similarly, P(Tails) = P(T) = $\frac{52}{100}$ .

Probabilities which are determined from experiments or from practical events are called **experimental probabilities** .

In general, the experimental probability of an outcome

$$= \frac{\text{the number of outcomes}}{\text{the total number of trials}}$$

This ratio is also called **relative frequency** .

Thus, $P(H) = \frac{\text{Number of heads}}{\text{Total number of tosses}}$

$P(T) = \frac{\text{Number of tails}}{\text{Total numbers of tosses}}$

Table 12.1 shows part of the results of a football competition.

P = Number of matches played

W = Number of matches won

D = Number of matches drawn

L = Number of matches lost

| Teams | P | W | D | L |
|---|---|---|---|---|
| Utala | 18 | 9 | 7 | 2 |
| Mumiad | 18 | 9 | 4 | 5 |
| Brewers | 18 | 5 | 9 | 4 |
| Shabina | 18 | 2 | 5 | 11 |

Table 12.1

In this case, each match is a **trial** . The outcome of each trial is either a win (W), a draw (D) or a loss (L). Thus the experimental probability of

1. Utala winning = $\frac{9}{18}$
2. Utala losing = $\frac{2}{18}$
3. Shabina winning = $\frac{2}{18}$

What is the probability of

(a) Mumiad drawing?

(b) Brewers winning?

(c) Shabina losing?

### *Example 12.1*

It has been found that the probability that Liz arrives at work on time is 0.2. How many times would you expect her to be on time in the next 20 days?

### *Solution*

Let the number of days Liz arrives on time in the 20 days be x. Then, the experimental probability of her being on time

$$= \frac{\text{Number of days she is on time}}{\text{Total number of days}} = \frac{x}{20}.$$

Since the probability that she is on time is 0.2, then $\frac{x}{20} = 0.2$

$\Rightarrow x = 0.2 \times 20$ days = 4 days.

## Exercise 12.1

1. Open a textbook. Write down the last digit of the right hand page. Do this 40 times. For example, if the book opens at pages 120 and 121, write down 1.
   (a) What is the probability of getting
      (i) 0
      (ii) 1
      (iii) 2
      (iv) 4?
   (b) What is the probability of getting a number less than 5?
   (c) What is the probability of getting an odd number?
2. Toss a bottle top 50 times. Each time record whether the bottle top faces up or down.
   (a) What is the probability of the bottle top facing up?
   (b) What is the probability of the bottle top facing down?
3. Toss two bottle tops together. Record whether both face down, both face up or one faces down while the other faces up. Do this 40 times. What is the probability that
   (a) both face down?
   (b) both face up?
   (c) one faces down and one faces up?
4. Toss two coins together. Record whether they show 2 heads (HH), 2 tails (TT) or 1 head and 1 tail (HT). Do this 40 times. What is the probability of obtaining
   (a) HH
   (b) TT
   (c) HT?
5. Open a textbook. Record the number of letters of the first and last words on a page. Do this 30 times. What is the probability that
   (a) a word has more than 5 letters?
   (b) a word has 4 letters?
   (c) a word has less than 4 letters?
6. In the past twenty days that Zikomo has been going to work, he has had a lift seven times, used a bus nine times and walked four times. What is the probability that
   (a) he gets a lift?

(b) he does not pay to get to work?

7. In the last season, it was found that the probability of Ranscom Football Club winning a match was $\frac{1}{8}$ . How many matches would you expect the team to win if they play 24 matches in the next season?
8. In 30 fights, a boxer has won 24 fights, drawn 2 and lost 4. What is the probability that
   (a) he wins in the next fight?
   (b) he does not win the next fight?
9. The probability that it rains on Christmas day in town X is 0.3. What is the probability that it will not rain on Christmas day in that town?
10. If the probability of the school's volleyball team winning is 0.4, how many games has the team possibly played if it has won 16 games?
11. (a) In Question 2, what do you get when you add the answers to (a) and (b) ?
   (b) In Question 3, what do you get when you add the answers to (a) , (b) and (c) ?
   (c) In Question 4, what do you get when you add your answers to (a) , (b) and (c) ?
   (d) In Question 8, what do you get when you add your answers to (a) and (b) ?
12. Toss a coin 20 times. Each time, record whether heads or tails show up. Find the experimental probability
   (a) P(H)
   (b) P(T)
   (c) P(H) + P(T)

   What do you notice in each case?

   Repeat the process when the coin is tossed
   (i) 40 times
   (ii) 60 times
   (iii) 80 times and
   (iv) 100 times

   Copy and complete Table 12.2 .

| No. of tosses | 20 | 40 | 60 | 80 | 100 |
|---|---|---|---|---|---|
| P(H) | | | | | |
| P(T) | | | | | |
| P(H) + P(T) | | | | | |

Table 12.2

What do you notice about the values of P(H) and P(T) as the number of tosses increases?

What do you notice about P(H) + P(T) in each case?

13. (a) Place 10 bottle tops (3 of drink A, 5 of drink B, 2 of drink C) in a bag. Mix them, then pick one at random and note which brand it is and then return it. Do this 20 times. Find the experimental probability

    (i) P(A)
    (ii) P(B)
    (iii) P(C)

    (b) Repeat part (a) using 10 bottle tops, all of drink A.

## Range of probability measure

In Question 13(b) of Exercise 12.1, you should have obtained P(B) = 0.

Since the bag did not have any drink B bottle tops, it is **impossible** to pick a drink B bottle top from it. Hence we say that picking a drink B bottle top from the bag is an **impossible** event. Thus,

> The probability of an impossible event is 0 (zero).

In Question 13(b) of Exercise 12.1, you should have obtained P(A) = 1.

Since the bag contains only drink A bottle tops, it is **certain** that whatever bottle top you pick it is of drink A. We say that picking a drink A bottle top is a **certain (or sure) event** .

Thus,

> If an event is certain (or sure), its probability is 1.

1. What is the probability that if you put a shirt in water it will get wet?
2. What is the probability that the sun will rise tomorrow?
3. What is the probability that if you kick a ball up, it will not come down?

Look at the answers you obtained in Exercise 12.1. Did you obtain any probability less than 0 or greater than 1?

You should have noticed that for any event, the probability lies between 0 and 1 (inclusive).

In general,

For any event A, $0 \leq P(A) \leq 1$

In Questions 11 and 12(c), you should have obtained the value 1 in each case.

In general,

If the only possible events are A and B, and when A takes place B does not, and vice versa, then $\mathbf{P(A) + P(B) = 1}$ .

$A'$ or ~A is used to denote that event A has not taken place (has not occured). Thus, $\mathbf{P(A) + P(A') = 1}$ .

Probabilities can be expressed either as fractions or as decimals.

## Theoretical probability

In this section, we see how certain probabilities can be found without experimenting.

In Question 12 of Exercise 12.1, you should have noticed that as the number of tosses increased, the values of P(H) and P(T) were each closer and closer to 0.5 (or $\frac{1}{2}$ ). If the number of tosses is very large, for all practical purposes we have $P(H) = P(T) = \frac{1}{2}$ (or 0.5).

When a fair coin is tossed, there are only two possible outcomes H or T. The symmetry of the coin tells us that “heads” and “tails” have equal chances of occuring. We say that they are **equally likely** . It follows that P(H) = P(T).

Since P(H) + P(T) = 1 (from the earlier section), then P(H) = P (T) = $\frac{1}{2}$ .

**Alternatively,**

All the outcomes are H, T.

Number of all outcomes = 2

Number of times H appears = 1

Number of times T appears = 1

$\therefore$ P(H) = $\frac{\text{Number of times H appears}}{\text{Total number of outcomes}} = \frac{1}{2}$

and P(T) = $\frac{\text{Number of times T appears}}{\text{Total number of outcomes}} = \frac{1}{2}$

Since these values can be obtained without tossing a coin, they are called **theoretical probabilities** .

In general, if N = number of all possible outcomes of A and n = number of times a particular outcome appears, then P(A) = $\frac{n}{N}$ .

Note that equally likely events (or outcomes) have equal probabilities.

### *Example 12.2*

A coin is tossed twice. Find the probability of obtaining

(a) 2 heads

(b) 1 head, 1 tail

(c) 2 tails

### *Solution*

If a coin shows heads in the first toss and heads in the second toss, we indicate the outcome as HH.

If the coin shows heads in the first toss and tails in the second toss, we indicate the outcome as HT, etc.

So, all the possible outcomes are HH, HT, TH, TT. Number of all possible outcomes = 4

Number of outcomes with two heads = 1

Number of outcomes with 1 head and 1 tail = 2

Number of outcomes with two tails = 1

Hence

(a) P(2 heads) = P(HH) = $\frac{1}{4}$

(b) P(1 head, 1 tail) = P(HT) = $\frac{2}{4} = \frac{1}{2}$

(c) P(2 tails) = P(TT) = $\frac{1}{4}$ .

**Note:** The same results would be obtained if two coins were tossed together.

## *Example 12.3*

A bag contains 10 bottle tops of which 3 are of drink A, 5 are of drink B and 2 are of drink C. If a bottle top is picked from the bag at random, what is the probability that it is

(a) a drink A

(b) a drink B

(c) a drink C?

## *Solution*

Let A represent drink A, B represent drink B and C represent drink C.

All the possible outcomes are

A, A, A, B, B, B, B, B, C, C.

Thus, the number of possible outcomes is 10. Each bottle top is equally likely to be picked.

Number of possible outcomes for drink A is 3.

Number of possible outcomes for drink B is 5.

Number of possible outcomes for drink C is 2.

(a) P(A)

$$= \frac{\text{No. of possible outcome for drink A}}{\text{No. of all possible outcomes}}$$

$$\text{i.e. } P(A) = \frac{\text{No. of drink A bottle tops}}{\text{No. of all bottle tops}} = \frac{3}{10}$$

(b) $P(B) = \frac{\text{No. of drink B bottle tops}}{\text{No. of all bottle tops}} = \frac{5}{10} = \frac{1}{2}$

(c) $P(C) = \frac{\text{No. of drink C bottle tops}}{\text{No. of all bottle tops}} = \frac{2}{10} = \frac{1}{5}$

(Compare these results with your answers for Question 13(a) of Exercise 12.1).

## Exercise 12.2

1. A box contains 12 boiled eggs and 25 raw ones. An egg is taken at random. What is the probability that the egg is raw?
2. A fair coin is tossed three times. Find the probability of getting
   (a) at least one heads.
   (b) two tails given that the first toss was heads.
3. If I choose a number at random from 11, 13, 15, 17, …………. 41, what is the probability that the number is either prime or a multiple of three?
4. A die is tossed once. What is the probability that the number appearing on top is prime?
5. A card is drawn from a pack of 52 playing cards. Find the probability that it is
   (a) an eight of clubs.
   (b) a king of diamonds.
6. In a room there are three gentlemen and four ladies. One person is picked at random. What is the probability that the person is a lady?
7. A vendor has fifteen 5 Kwacha coins and nine 10 Kwacha coins in a bag. He picked one coin from the bag at random. What is the probability that the coin was a 10 Kwacha coin?
8. In a class of 20, there were 8 boys. If one student was picked at random, what is the probability that a girl was picked?

9. A set of cards are numbered from 1, 2, ….30. One card is picked at random. What is the probability that the number of the card is a
   (a) multiple of 3?
   (b) factor of 28?
10. In 2011, a national soccer team played against four other national teams. If the team won once and drew three times. What is the probability that the team will win in 2012?

## 6-12 REVISION EXERCISES 2

### Revision exercise 2.1

1. Use graphical method to solve the simultaneous equations.

   $3x + y = 17$

   $2x + 3y = 23$

2. Given that $x$ and $y$ are two numbers such that their sum is 48 and their difference is 12. Find the numbers.

3. Simplify

   (a) $5^4 \times 5^{15}$
   (b) $7^{11} \times 7^{20}$
   (c) $a^{-6} \times a^{8}$
   (d) $c^{-6} \times c^{-4}$
   (e) $x^{3} \div x^{-8}$
   (f) $y^{5} \div y^{-3}$
   (g) $y^{-7} \div y^{2}$
   (h) $d^{-11} \div d^{-13}$

4. (a) Express the following in standard form.

   (i) 68 700 000
   (ii) 0.000 000 142 7

   (b) Given that $a = 2.5 \times 10^9$ and $b = 1.5 \times 10^{-7}$ , evaluate (i) $ab$ (ii) and give your answers in standard form.

   (c) Evaluate $0.68 \times 1.2 \times 10^{-4} \times 1\,000 \times 2 \times 10^{11}$ and give your answer in standard form.

5. Given that $y$ varies directly as $x$ and that $y = 6$ when $x = 4$, calculate

   (a) the value of $y$ when $x = 6$
   (b) the value of $x$ when $y = 4$

6. X(2, 2), Y(4, 2) and Z(3, 4) are the vertices of a triangle. The triangle is enlarged by a scale factor 2 and centre origin. What are the coordinates of the vertices of the image ΔX´Y´Z´?

7. Which of the following inequalities are true?

(a) $^{-}3 < {}^{-}2$
(b) $3(^{-}1) > 2(^{-}1)$
(c) $2 < {}^{-}3$
(d) $4(2) < 5(2)$
(e) $(^{-}2)(^{-}1) > 3(^{-}1)$
(f) $4(^{-}2) < 5(^{-}2)$

Correct those that are false.

8. Find the range of values of $x$ which satisfy the inequality $2x + 1 < 11 < 6x - 1$.
Illustrate your answer on a number line.

9. A newspaper vendor had three papers: The Nation, Daily Times and Nyasa Times, to deliver to three people A, B and C, respectively. He forgot the list of who was to receive which paper and so he delivered the papers at random. What is the probability that
(a) everyone received the correct paper?
(b) only one person got the correct paper?
(c) no one got the correct paper?

10. A coin is biased so that the probability of "heads" is $\frac{3}{4}$. Find the probability that when the coin is tossed three times, it shows
(a) 3 tails
(b) 2 heads and 1 tail
(c) no tail

11. A passanger train 160 m long travels at a speed of 260 km/h. A goods train 148m long travels at 25 km/h in the same direction. How long would the passanger train take to overtake the goods train completely?

12. A cyclist starts from town A at 12.00 noon and rides towards town B at a speed of 12 km/h. At 2.00 p.m, a motorist leaves town B and travels towards town A at a speed of 36 km/h. Given that the two towns are 60 km apart, show the two journeys on the same graph. Use your graph to find
(a) the time when the cyclist meets the motorist and how far from town A.
(b) the time the motorist and cyclist are 12 km apart.

## Revision exercise 2.2

1. Solve the simultaneous equations.

   (a) $3q - p = 6$
   $-10q + 3p = 15$

   (b) $m + \frac{1}{2}n = 13$
   $\frac{1}{3}m - n = 2$

   (c) $\frac{2x}{5} - \frac{y}{3} = 2\frac{2}{3}$
   $x = 2(y + 1)$

   (d) $3x + \frac{1}{2}y = 8x + 7y - 9 = 2$

2. Mr. Ndoya has exactly K 700 in his wallet. This money is composed of K 50 notes and K 100 notes. If he has a total of 11 notes, how many notes of each kind does he have?

3. (a) Simplify the following, giving your answers in the same bar notation.

   (i) $\bar{3}.2 + 1.4$
   (ii) $\bar{6}.3 - 1.7$
   (iii) $\bar{4}.621 \times 2$
   (iv) $\bar{2}.881 \div 3$

   (b) Find antilogarithms of

   (i) 0.765 2
   (ii) 1.358 7
   (iii) $\bar{2}.666\,6$
   (iv) 0.007 8

   (c) Evaluate

   (i) $10^{0.3}$
   (ii) $10^{1.257\,6}$
   (iii) $10^{2.912}$
   (iv) $10^{\bar{3}.687\,1}$
   (v) $10^{\bar{1}.002\,5}$
   (vi) $10^{\bar{2}.163\,2}$

4. Write the following using positive indices:

   (a) $12^{-2}$
   (b) $a^{-7}$
   (c) $2x^{-4}$

(d) $\frac{a^2}{b^{-4}}$

(e) $\frac{x^{-3}}{w^{-4}}$

(f) $\frac{a^{-5}}{b^{-10}}$

5. Write each of the following ratios in the simplest form.

(a) 16 : 24

(b) 24 : 21

(c) 50 : 60 : 70

(d) 20 : 25: 30 : 35

(e) $\frac{3}{4} : \frac{1}{3}$

(f) $\frac{1}{2} : \frac{3}{8}$

(g) $2\frac{1}{4} : \frac{1}{4}$

(h) $1\frac{1}{3} : 3\frac{1}{4}$

6. Triangles PQR and XYZ are similar. PQ = 24 cm, QR = 32 cm, RP = 36 cm and YZ = 8 cm. What are the lengths of the remaining sides of ΔXYZ?

7. Write down the inequalities illustrated in Fig. R2.1 .

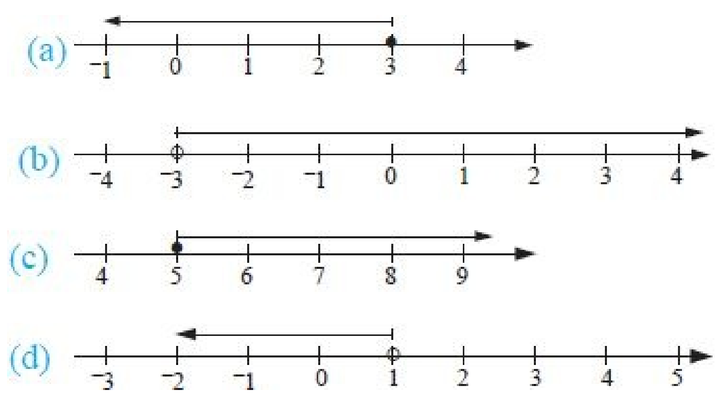



*Fig. R2.1*

8. The following pairs of inequalities are equivalent. True or false?

(a) $3x > 5;\ x > \frac{5}{3}$

(b) $2 - x > 4;\ x > {}^{-}2$

(c) $\frac{1}{2}x < 6;\ {}^{-}2x > {}^{-}24$

(d) $x - 3 \geq 5;\ x > 8$

9. A basket contains 48 ripe mangoes and 100 raw ones. A mango is taken at random. What is the probability that the mango is raw?

10. (a) A whole number from 1 to 40, inclusive, is selected. If each number has the same chance of being selected, what is the probability that the number is prime?
(b) A fair die is thrown 60 times. How many times is
    (i) a 1 expected to show up?
    (ii) a number divisible by 3 expected to show up?

11. Table R1.1 shows the distance covered by a cyclist in a time t (s).

| Time (s) | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| Distance (m) | 0 | 12.5 | 30 | 50 | 72.5 | 100 |

*Table: R1.1*

(a) Draw a distance-time graph for this motion.
(b) Use your graph to determine the speed of the cyclist for the first 5 s.

## Revision exercise 2.3

1. Use elimination method to solve the simultaneous equations.
    (a) $x + 2y = 11$
    $2x - y = 2$
    (b) $3x - y = 11$
    $2x - 3y = 5$

2. Use substitution method to solve the simultaneous equations.
    (a) $x - 2y = 27$
    $7x + y = 9$
    (b) $4x = 3y + 2$
    $3x + y + 1 = 0$

3. Evaluate the following:
    (a) $81^{\frac{-1}{4}}$
    (b) $32^{\frac{-2}{5}}$
    (c) $\left(\frac{243a^6 \times 2b^3}{18}\right)^{\frac{1}{3}}$

(d) $\dfrac{(3a^6)^{\frac{-1}{3}}}{(243)^{\frac{1}{3}}a}$

(e) $\dfrac{6q^{\frac{1}{4}} \times q^{\frac{2}{3}}}{18q^{\frac{1}{12}}}$

4. (a) State the characteristic of each of the logarithms of the following numbers.

   (i) 3.5
   (ii) 27.2
   (iii) 4 763
   (iv) 18 002
   (v) $10^{2.5}$
   (vi) 0.003 4

   (b) State the logarithms of the following:

   (i) 5.23
   (ii) 45.62
   (iii) $10^{3.6}$
   (iv) $4 \times 10^5$
   (v) 0.001 389 7

5. If $b$ varies inversely as $e$ , and $b = 6$ when $e = 2$, calculate

   (a) the value of $b$ when $e = 12$.
   (b) the value of $e$ when $b = 3$.

6. On squared paper, plot A(4, 5), B(4, 3) and C(8, 1) and draw ΔABC. Plot also A´(2, 3) and B´(2, 2), the images of A and B under a certain enlargement.

   By appropriate construction locate point C´, a vertex of ΔA´B´C´. State the coordinates of C´ and find the scale factor of enlargement.

7. Solve the following inequalities.

   (a) $^{-}2 + 3 \leq {}^{-}5x$
   (b) $^{-}4 + 7 < {}^{-}2x - 5$
   (c) $2x - 7 < 32 - x$
   (d) $(5x - 4) \leq x + 5$

8. Represent each of the following inequalities on a number line.

   (a) $x > 3$

(b) $x < 4$
(c) $x \geq 2$
(d) $x \leq {}^{-}4$

9. Gumurila is learning how to play darts. The probability that he hits the mark when he throws a dart is $\frac{1}{10}$. If he tries four times, find the probability that he
   (a) hits the mark four times.
   (b) does not hit the mark at all.
   (c) hits the mark at least once.

10. All possible two-digit numbers are formed from the digits 1, 2, 3, 4, 5, 6, 7. If one of these numbers is chosen at random, what is the probability that it is divisible by
    (a) 5?
    (b) 3?

11. At 2000 h a bus P leaves town A travelling to town B, 75 km away at a speed of 60 km/h. After a 30 min stopover at B, due to the poor state of the road, the bus travels another 35 km to town C at 35 km/h. At 2100 h, an express bus Q leaves town C for A, and travels at a speed of 55 km/h.
    (a) Draw the distance-time graph to show both journeys.
    (b) From your graph, find where and when the two buses met.
    (c) What time did bus P and bus Q reach their respective destinations?

12. A man took 3 hours to travel from Town A to Town B driving at a speed of 140 km/h. Without stopping, he continued to town C, driving at a speed of 110 km/h and took 2 hours. Calculate the distance between Town A and Town C.

# 13 QUADRATIC EQUATIONS I

## Quadratic equations

In Form 1, we studied linear equations and their solutions. Now, we shall consider the solution of an equation that contains the unknown to the second power. Such an equation is called a **quadratic equation** .

Quadratic equations can be written in the form $ax^2 + bx + c = 0$ where $a$, $b$ and $c$ are constants, and $a \neq 0$.

The form $ax^2 + bx + c = 0$ is called the **standard quadratic form** of a quadratic equation.

When the equation is in this standard form, the terms on the LHS are arranged in descending powers of the variable. On the RHS, there is only zero.

When solving quadratic equations, it is necessary to write the equation in the standard form.

The solution of a quadratic equation in the standard form is the value of the variable that makes the LHS equal to zero, hence, forming a true statement as we did with linear equations.

## Solving quadratic equations

Consider the equation $x^2 + 5x + 6 = 0$.

In factor form, the equation becomes $(x + 2)(x + 3) = 0$.

This equation states that the product of $(x + 2)$ and $(x + 3)$ is zero.

We can solve this equation only if we know the following fact.

> If $a$ and $b$ are real numbers, and $a \times b = 0$, then either $a = 0$ or $b = 0$ or $a = b = 0$.

By this property, if $(x + 2)(x + 3) = 0$, then either $x + 2 = 0$ or $x + 3 = 0$.

Since each of these factors is linear, we use the method of solving linear equations.

If $x + 3 = 0$ and if $x + 2 = 0$

then $x = {}^-3$ then $x = {}^-2$

$\therefore {}^-3$ and ${}^-2$ are the solutions of the equation $(x + 3)(x + 2) = 0$.

From the above discussion, we can see that to solve any equation, in factored form, whose product is 0, we

1. set each factor equal to zero,
2. solve the resulting equations for the variable.

***Example 13.1***

*Solve the equation (a) $(3x + 4)(2x - 1) = 0$*

*(b) $x(x - 5) = 0$*

***Solution***

*(a) $(3x + 4)(2x - 1) = 0$*

*(For the product to be equal to zero, one or the other of the factors must be equal to 0).*

*If $3x + 4 = 0$ or if $2x - 1 = 0$*

*then $3x = {}^-4$ then $2x = 1$*

$\therefore x = \frac{-4}{3}$ $\therefore x = \frac{1}{2}$

*The solution of the equation*

*$(3x + 4)(2x - 1) = 0$ is $x = \frac{-4}{3}$ or $x = \frac{1}{2}$.*

*(b) $x(x - 5) = 0$. This equation has two factors, $x$ and $x - 5$.*

*Either $x = 0$ or $x - 5 = 0$*

*$\therefore x = 0$ or $5$ is the solution of the equation $x(x - 5) = 0$*

**Note:** The solutions of a quadratic equation are also known as its **roots** .

## Exercise 13.1

Solve the following following equations.

1. (a) $x + 6 = 0$

   (b) $x - 9 = 0$

2. (a) $3x + 9 = 0$
   (b) $6 – 4x = 0$
3. (a) $5x – 7 = 0$
   (b) $(x + 5)(x – 5) = 0$
4. (a) $(x – 1)(x + 2) = 0$
   (b) $x (x + 6) = 0$
5. (a) $5p (p + 9) = 0$
   (b) $x (x – 8) = 0$
6. (a) $(3x – 9)(2x + 3) = 0$
   (b) $(2x – 1)(x + 2) = 0$
7. (a) $(2x + 1)(3x – 2) = 0$
   (b) $(4x – 1)(3x + 1) = 0$
8. (a) $(4 – 3a )(8 – 5a ) = 0$
   (b) $(5 – y )(y – 7) = 0$
9. (a) $(2x + 7)(x – 3) = 0$
   (b) $(5x + 3)(x – 10) = 0$
10. (a) $(x – 10)(4x – 3) = 0$
    (b) $3x (x – 12) = 0$
11. (a) $(x – 12)(3x + 1) = 0$
    (b) $3(2x + 11) = 0$
12. (a) $(8x – 1)(2x + 7) = 0$
    (b) $(7 + 3y )(2 – 3y ) = 0$

## Solving quadratic equations involving perfect squares

From chapter 3 on Algebraic Processes II, recall,

1. $(a + b )^2 = a^2 + 2ab + b^2$
2. $(a – b )^2 = a^2 – 2ab + b^2$

These expressions are referred to as perfect squares.
To solve quadratic equations involving perfect squares we proceed as follows.

1. Express the equation in standard form $ax^2 + bx + c = 0$
2. Factorise the left hand side (LHS) of the equation.
3. Set the repeated factor equal to zero once.
4. Solve the linear equation.

Consider the following examples.

### *Example 13.2*

*Solve the equations* $a^2 + 14a + 49 = 0$.

### *Solution*

$a^2 + 14a + 49 = 0$

$(a + 7)^2 = 0$ *(Factorising the LHS)*

$(a + 7) = 0$ *(Setting repeated factor equal to 0 once)*

*Solving the linear equation, we get,*

$a = {}^{-}7$

*Check:* $({}^{-}7)^2 + 14 \times {}^{-}7 + 49 = 0$

$49 - 98 + 49 = 0$

$0 = 0$

### *Example 13.3*

*Solve the equation* $4x^2 = 20x - 25$

### *Solution*

*Write the equation in standard form*

$4x^2 = 20x - 25 \Rightarrow 4x^2 - 20x + 25 = 0$

$(2x - 5)^2 = 0$ *(Factorising the LHS)*

$2x - 5 = 0$ *(Setting the repeated factor equal to zero once)*

$2x = 5$

$\therefore x = \frac{5}{2}$ *(Solving the linear equation) Check the solution by substituting* $\frac{5}{2}$ *for x in the original equation.*

**Note:** In these examples, we have two identical factors. In each case, since the two factors are the same, we need to solve the linear equations only once, and we say the equation has only one **distinct** solution. This is an example of a **repeated root** . It should be written as

x = $\frac{5}{2}$ **twice** .

## Exercise 13.2

Solve the following equations.

1. $x^2 + 6x + 9 = 0$
2. $x^2 – 2x + 1 = 0$
3. $y^2 = 28y + 16$
4. $x^2 + 10x = {}^{-}25$
5. $4y^2 = 28y – 49$
6. $4p^2 + 36p = {}^{-}81$
7. $n^2 + n + \frac{1}{4} = 0$
8. $a^2 – 3a + \frac{4}{9} = 0$

## Solving quadratic equations by factor method

We have just seen that if the LHS of an equation is a product of linear factors and the RHS equals zero, there will be one solution of the equation for each linear factor. Thus, we can solve some standard quadratic equations of the form $ax^2 + bx + c = 0$ by factorising the left hand side of the expression into linear factors.

In general, to solve a quadratic equation by factor method, we use the following procedure.

1. Write the equation in standard quadratic form.
2. Factorise the left hand side completely.

3. Set each of the factors containing the variable equal to zero and solve the resulting equations.
4. Check the solutions by substituting into the original equation. (Substitute one solution at a time).

**Note** that a quadratic equation must have two roots.

### *Example 13.4*

*Solve the quadratic equation* $x^2 - 2x - 8 = 0$.

### *Solution*

$x^2 - 2x - 8 = (x - 4)(x + 2)$

*Thus,* $x^2 - 2x - 8 = 0$ *becomes* $(x - 4)(x + 2) = 0$

$\Rightarrow$ *Either* $x - 4 = 0$ *or* $x + 2 = 0$

$\therefore x = 4$ *or* $x = {}^{-}2$

*The solution of* $x^2 - 2x - 8 = 0$ *is* $x = 4$ *or* $x = {}^{-}2$.

## Exercise 13.3

Solve the following equations.

1. (a) $x^2 + 3x + 2 = 0$
   (b) $x^2 - 14x + 30 = 0$
2. (a) $x^2 + 6x - 16 = 0$
   (b) $y^2 - 3y - 4 = 0$
3. (a) $x^2 = {}^{-}11x - 10$
   (b) $a^2 - 11a = 12$
4. (a) $x^2 - 14x = 15$
   (b) $x^2 - 32 = 4x$
5. (a) $x^2 = 27 - 6x$
   (b) $a^2 - 3a = 28$

6. (a) $b^2 - 7b = {}^-10$

   (b) $y^2 - y = 6$

7. (a) $x^2 + 4x = 0$

   (b) $a^2 - 4a = 0$

8. (a) $3y^2 - 5y = 0$

   (b) $4y^2 + 7y = 0$

9. (a) $2x^2 + 6x = 0$

   (b) $3x^2 = 9x$

10. (a) $2x^2 - 18 = 0$

    (b) $7x^2 - 7 = 0$

11. (a) $x^2 - 100 = 0$

    (b) $2x^2 - 7x - 9 = 0$

12. (a) $6x^2 - 5x + 1 = 0$

    (b) $9x^2 + 20 = {}^-27x$

13. (a) $6z^2 + z = {}^-10z - 3$

    (b) $3x^2 - 4x - 28 = x$

14. (a) $3a^2 - 3 = 5a^2 + 3a - 5$

    (b) ${}^-6x = {}^-3x^2 - 3$

15. (a) $x(x + 3) = {}^-2$

    (b) $2x(2x + 6) = {}^-8$

## Equations leading to quadratic equations

Many equations involving fractions eventually lead to quadratic equations.

***Example 13.5***

*Solve the equation* $\dfrac{3}{2x+1} + \dfrac{4}{5x-1} = 2.$

***Solution***

*Step 1: Multiply each term by the LCM of the denominators in order to remove the fractions:*

*LCM of denominators = (2x + 1)(5x – 1)*

$\therefore 2(2x + 1)(5x - 1)\left(\frac{3}{2x+1} + \frac{4}{5x-1}\right)$

$= (2x + 1)(5x - 1)$

$(5x - 1) \times 3 + (2x + 1) \times 4$

$= (2x + 1)(5x - 1) \times 2$

*Step 2: Multiply out and simplify:*

$3(5x - 1) + 4(2x + 1)$

$= 2(2x + 1)(5x - 1)$

$15x - 3 + 8x + 4$

$= 2(10x^2 - 2x + 5x - 1)$

$23x + 1 = 20x^2 + 6x - 2$

*Step 3: Rearrange the equation in the standard quadratic form:*

$20x^2 + 6x - 2 - 23x - 1 = 0$

$20x^2 - 17x - 3 = 0$

*Step 4: Factorise and solve:*

$20x^2 - 20x + 3x - 3 = 0$

$20x(x - 1) + 3(x - 1) = 0$

$(x - 1)(20x + 3) = 0$

$x - 1 = 0$ or $20x + 3 = 0$

$\therefore x = 1$ or $x = -\frac{3}{20}$

## Exercise 13.4

Solve the following equations

1. $y + \frac{1}{y} = 2\frac{1}{2}$
2. $\frac{x+2}{x-2} = \frac{x+3}{x-9}$
3. $\frac{1-x}{2} = \frac{1}{9x}$

4. $2 - \frac{1}{x} = x$

5. $\frac{3x+5}{6x+5} = x - 1$

6. $\frac{2x^2}{x+1} + 1 = \frac{2}{x+1}$

7. $x - \frac{4}{x} + 3 = 0$

8. $x + \frac{1}{x} = {}^{-}2$

9. $x - \frac{9}{x} = 0$

10. $x - 1 = \frac{1}{x-1}$

11. $(x+5) + \frac{18}{x+5} - 9 = 0$

12. $x - \frac{35}{x+2} = 0$

13. $x - 9 = \frac{72}{x-8}$

14. $\frac{x+10}{x-5} = \frac{7x}{x-5}$

## Word problems leading to quadratic equations

Many word problems require the use of quadratic equations for their solution.

### *Example 13.6*

*The product of two consecutive even numbers is 168. Find the numbers.*

### *Solution*

*Let x be the smaller even number and x + 2 be the larger. (Consecutive even numbers differ by 2).*

*From 'the product of two consecutive even numbers', we get x (x + 2).*

$\therefore x^2 + 2x = 168$

$x^2 + 2x - 168 = 0$ *(Rearranging in standard form)*

$(x + 14)(x - 12) = 0$ *(Factorising the LHS)*

$\therefore x + 14 = 0$ or $x - 12 = 0$ *(Setting each factor equal to zero and solving)*

*i.e.* $x = {}^{-}14$ or $x = 12$

*When $x = {}^{-}14$, the next even number $x + 2 = {}^{-}12$.*

*When $x = 12$, the next even number $x + 2 = 14$.*

*The consecutive even numbers are ${}^{-}14$ and ${}^{-}12$ or 12 and 14.*

*Check:* ${}^{-}14 \times {}^{-}12 = 168$

and $12 \times 14 = 168$.

*Both pairs give consecutive even numbers and the conditions of the problem are met.*

## *Example 13.7*

*The length of a rectangle is 2 m longer than the breadth. If the area of the rectangle is 80 $m^2$, find its dimensions.*

## *Solution*

*Let $w$ be the width and $l$ the length of the rectangle (see Fig. 13.1 )*

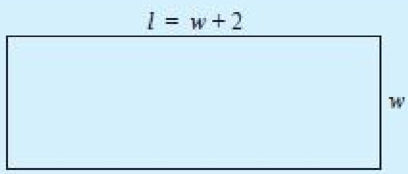



*Fig. 13.1*

$$\text{Now, Area } (A) = l \times w$$

$$\text{so, } w \times l = 80$$

$$w(w + 2) = 80$$

$$w^2 + 2w = 80$$

$$w^2 + 2w - 80 = 0$$

$$(w + 10)(w - 8) = 0$$

$$\therefore w = {}^{-}10 \text{ or } 8.$$

*Since the width of a rectangle cannot be negative, $w = {}^{-}10$ is not a solution of the problem.*

*$\therefore w = 8$ is the only practical solution of the problem.*

*$l = w + 2 = 8 + 2 = 10$ m.*

*The rectangle is 8 m wide and 10 m long.*

## Exercise 13.5

1. One integer is 3 less than another. If their product is 340, find the integers.
2. The length of a rectangle is double its breadth. If its area is 200 $m^2$ , find its length.
3. The product of two consecutive whole numbers is 42. Find the numbers.
4. Find a number such that the sum of the number and its reciprocal is $4\frac{1}{4}$ .
5. A rectangle is of length $(x + 2)$ m and width $(x - 1)$ m. Write an expression for its area in the simplest form.
6. The sum of the squares of two consecutive integers is 113. Find these integers.
7. Three times the square of a certain number is decreased by 9 times the number. The result is 120. Find the number.
8. Think of a number, square it and add the original number. If your result is 56, what is the number?
9. Think of a number, multiply it by 3, subtract 1 and square the result. If this result is the same as the square of the original number, find the number.
10. Find two consecutive odd numbers such that the sum of their squares is 650.
11. In Fig. 13.2 , all the dimensions are in centimetres, and all the angles are right angles. If the area of the figure is 90 $cm^2$ , find the value of $x$ .

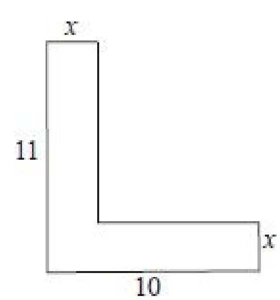



*Fig. 13.2*

12. An $n$– sided figure has $\frac{1}{2}n(n-3)$ diagonals. How many sides has the figure if it has 135 diagonals?

13. ABC is a right-angled triangle (Fig. 13.3 ) and its dimensions are given in centimetres.

Find the lengths of the sides of the triangle.

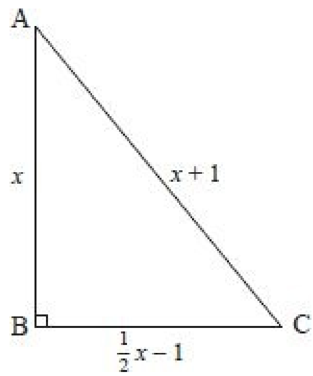



*Fig. 13.3*

# 14 DENSITY AND MIXTURES

## Density

Consider two different solids such as a wooden block and a building stone of the same size (same volume). The two solids are lifted. Which one is heavier and why?

The building stone is heavier than the wooden block because they are made of different materials which have different *masses* in equal *volumes*, hence different *densities.*

**Mass** is the quantity of matter in a substance.

The mass of an object is found by comparing it with a standard mass using a beam balance.

The **gram** or **gramme** (abbreviated as g) is the most common unit of mass. One gram is the mass of 1 $cm^3$ (1 ml) of water at a temperature of 4°C. The SI unit of mass is the **kilogram** or **kilogramme** (kg).

The **density** of a substance is the mass of a unit volume of that substance. It is given by the formula:

Density = $\frac{\text{Mass}}{\text{Volume}}$ or $D = \frac{m}{V}$

If mass is in g and volume is in $cm^3$ , then density is measured in $g/cm^3$ .

If the mass is in kg and volume is in $m^3$ , then density is measured in $kg/m^3$ .

To convert density from $g/cm^3$ to $kg/m^3$ we multiply by 1 000, while from $kg/m^3$ to $g/cm^3$ we divide by 1 000.

### *Example 14.1*

*If the density of methylated spirit is 0.8 g/ $cm^3$ , what is the volume, in $m^3$ , of 4 800 kg of methylated spirit?*

### *Solution*

Density = $\frac{Mass}{Volume}$

Density = $0.8\ g/cm^3$

$1\ m^3$ ($=1\ 000\ 000\ cm^3$) has a mass of $0.8 \times 1\ 000\ 000\ g$

So, $1\ m^3$ has a mass of $\frac{0.8 \times 1\ 000\ 000}{1\ 000}\ kg$

$= 800\ kg$

So, density $= 800\ kg/m^3$

$800\ kg/m^3 = \frac{4\ 800}{Volume}$

Volume $= \frac{4\ 800}{800} m^3$

$= 6\ m^3$

## Exercise 14.1

1. Express the following in:

   (a) $g/cm^3$

   (i) $1.5\ kg/m^3$
   (ii) $18\ kg/m^3$
   (iii) $300\ kg/m^3$
   (iv) $4\ 500\ kg/m^3$
   (v) $27\ 500\ kg/m^3$

   (b) $kg/m^3$

   (i) $12\ g/cm^3$
   (ii) $250\ g/cm^3$
   (iii) $500\ g/cm^3$
   (iv) $0.56\ g/cm^3$
   (v) $0.08\ g/cm^3$
   (vi) $0.007\ 5\ g/cm^3$

2. Find the densities of each of the following:

   (a) A block of wood of volume $730\ cm^3$ and a mass of 512 g.
   (b) A cube of gold of mass 1 235 g and volume of $64\ cm^3$ .

(c) Steel of mass 43.68 kg and volume of 0.005 6 $m^3$ .

3. The densities of petrol, glass, lead and mercury are 0.8 $g/cm^3$ , 2.5$g/cm^3$ , 11.4 $g/cm^3$ and 13.6 $g/cm^3$ respectively. Find the
   (a) volume of 6 500 kg of petrol.
   (b) mass of 40 litres of petrol.
   (c) mass of 1.5 $m^3$ of lead.
   (d) mass of 0.5 $m^3$ of mercury.
   (e) mass of glass that has the same volume as 10 500 kg of petrol.
   (f) mass of lead that has the same volume as 5 kg of mercury.
4. A pipe has an external diameter of 5.6 cm and internal diameter of 4.9 cm. Find the mass of a length of 15 m of the pipe if the material is of density 7.6 $g/cm^3$ .
5. A rectangular block is 50 cm long and 15 cm wide. If its mass is 18 kg and its density is 2.4 $g/cm^3$ , find its height.
6. The density of ice is 900 $kg/m^3$ . What is the volume of a block of ice of mass
   (a) 55 g
   (b) 4.8 g
   (c) 2.3 kg
   (d) 36 kg?
7. The density of glass is 2.5 $g/cm^3$ . What is the mass of glass of volume
   (a) 1 $cm^3$
   (b) 8 $cm^3$
   (c) 22.5 $cm^3$
   (d) 0.55 $cm^3$ ?
8. The length of a hall is 18.2 m, its width is 6.5 m and its height is 3.3 m. If the density of air is 1.3 $kg/m^3$ , what is the mass of the air in the hall?

## Relative density

It is sometimes necessary to compare densities of substances. To do so, the density of a substance is compared to the density of a known substance. This comparison is known as **relative density** . Water being the most common and with a density of 1 $g/cm^3$ is used as the reference substance.

*Thus, Relative density* = $\dfrac{\textit{density of a substance}}{\textit{density of water}}$

When the volume of a substance and that of water are equal, the relative density is given by:

$$\textit{Relative density} = \frac{\textit{mass of a substance}}{\textit{mass of an equal volume of water}}$$

In general,

> The ratio of the mass of any volume of a substance to the mass of an equal volume of water (at 4ºC) is called the **relative density** of that substance.

Since relative density is a ratio, it has no units.

## *Example 14.2*

*If the density of water is 1 $g/cm^3$ , find the relative densities of the following substances.*

*(a) A cylindrical copper wire of density 8.936 $g/cm^3$ .*

*(b) Paraffin of density 180 $kg/m^3$ .*

## *Solution*

*Relative density (R.D)* = $\dfrac{\textit{density of substance}}{\textit{density of water}}$

*(a) R.D* $= \dfrac{\textit{density of copper}}{\textit{density of water}}$

$= \dfrac{8.936\ g/cm^3}{1\ g/cm^3} = 8.936$

*(b) Density of paraffin in $g/cm^3$ = 0.18 $g/cm^3$ R.D*

$$= \frac{\text{density of paraffin}}{\text{density of water}}$$

$$= \frac{0.18}{1} = 0.18$$

***Example 14.3***

*A rectangular tin measuring 16 cm by 12 cm, contains petrol of relative density 0.68 to a depth of 40 cm. Find to 2 s.f. the mass of the petrol in kg.*

***Solution***

$$R.D = \frac{\text{density of petrol}}{\text{density of water}}$$

$\therefore$ *density of petrol* $= 0.68 \times 1\ g/cm^3$

$= 0.68\ g/cm^3$

*Volume of petrol* $= 16\ cm \times 12\ cm \times 40\ cm$

$= 7\ 680\ cm^3$

*Mass of petrol = density × volume*

$= 0.68 \times 7\ 680$

$= 5\ 222.4\ g$

$= 5.2\ kg\ (2\ s.f.)$

From Examples 14.2 and 14.3, notice that the density of a substance in $g/cm^3$ is numerically equal to its relative density. Table 14.1 shows the densities and relative densities of some common substances.

| **Substance** | **Density ($g/cm^3$ )** | **Relative density** |
|---|---|---|
| Air | 0.001 293 | 0.001 293 |
| Petrol | 0.68 | 0.68 |
| Sea water | 1.03 | 1.03 |
| Paraffin | 0.18 | 0.18 |
| Mercury | 13.6 | 13.6 |
| Brick | 1.3 | 1.3 |

| | | |
|---|---|---|
| Glass | 2.5 | 2.5 |
| Aluminium | 2.7 | 2.7 |
| Iron | 7.5 | 7.5 |
| Brass | 8.5 | 8.5 |
| Copper | 8.93 | 8.93 |
| Steel | 7.8 | 7.8 |

*Table 14.1*

## Exercise 14.2

Take density of water as 1 g/cm$^3$ .

1. A metal whose volume is 1 m$^3$ , has a mass 516 kg. What is its relative density?
2. A metallic block 4 cm long, 3 cm wide and 1.5 cm thick has a mass of 185.4 g. Find the relative density of the metal.
3. Find the mass of a rectangular block of rock 40 cm long, 35 cm wide and 36 cm thick given that the relative density is 2.6.
4. A solid of relative density 1.2 has a mass of 62.4 g. Find its volume in cm$^3$ .
5. An iron bar, relative density 7.75 has a circular cross-section of diameter 3.5 cm and is 1 m long. Find its mass.
6. A cylindrical wooden stick 92 cm long and 1.8 cm diameter weighs 170 g. What is the relative density of the wood?
7. A silver cylinder is 6 cm high and the diameter of its base is 3 cm. What is its mass if its relative density is 10.5?
8. A flask has a mass of 23 g when empty, 140 g when full of water and 102 g when full of another liquid. Find the relative density of the liquid.
9. A container has a mass of 7 g when empty, 18 g when full of water and 17.1 g when full of oil. Find the relative density of the oil.
10. A cylindrical tank 3 m high and 2 m in diameter is full of petrol whose relative density is 0.68. Find the mass of the petrol.

## Density of mixtures

Occasionally, two or more substances are mixed in a given proportion. The density of the mixture is found from the density of the substances used to make it.

### *Example 14.4*

*1 kg of sugar, density 1.1 g/cm$^3$ and 0.25 kg of salt, density 1.2 g/cm$^3$ , are mixed together for a certain experiment. What is the density of the mixture? (Give the answer to 4 s.f.)*

### *Solution*

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}} \qquad \therefore \text{Volume} = \frac{\text{Mass}}{\text{Density}}$$

*Total mass = 1 000 g + 250 g = 1 250 g*

$$\text{Volume of sugar} = \frac{1\ 000\ \text{g}}{1.1\ \text{g/cm}^3} = 909.1\ \text{cm}^3$$

$$\text{Volume of salt} = \frac{250\ \text{g}}{1.2\ \text{g/cm}^3} = 208.3\ \text{cm}^3$$

$\therefore$ *Combined volume = 1 117.4 cm$^3$*

$$\text{Combined density} = \frac{\text{Total mass}}{\text{Total volume}}$$

$$= \frac{1\ 250\ \text{g}}{1\ 117.4\ \text{g/cm}^3}$$

*= 1.119 g/cm$^3$ (to 4 s.f.)*

## Exercise 14.3

1. Three litres of water (density = 1 g/cm$^3$ ) is added to 12 litres of alcohol (density = 0.8 g/cm$^3$ ). What is the density of the mixture?
2. An alloy is made up of two metals of density 18 g/cm$^3$ and 10 g/cm$^3$ . Find the density of the alloy if
   (a) equal volumes of the metals were used.
   (b) equal masses of the metals were used.

3. 100 $cm^3$ of water of density 1 $g/cm^3$ is mixed with 120 $cm^3$ of alcohol of density 0.8 $g/cm^3$ . Find the density of the mixture.
4. Copper of volume 100 $cm^3$ and density 6.5 $g/cm^3$ is mixed with aluminium of volume 110 $cm^3$ and density 2.7 $g/cm^3$ . Find the density of the mixture.
5. 170 g of brass (density = 8.5 $g/cm^3$ ) is mixed with 25 $cm^3$ of iron (density = 7.5 $g/cm^3$ ). What is the density of the mixture?
6. The density of a mixture of sea water and fresh water is 1.1 $g/cm^3$ . If the mixture contains 20 $cm^3$ of sea water and 20 g of fresh water whose volume is 20 $cm^3$ , find
   (a) the mass of sea water in the mixture.
   (b) the density of sea water.
7. 200 kg of invar of density 8 $g/cm^3$ is mixed with 450 kg of copper of density 9 $g/cm^3$ .
   (a) Find the total volume of the mixture.
   (b) Find the density of the mixture.
8. 80 g of a liquid is mixed completely with 120 g of water. If the relative density of the liquid is 1.20, find the relative density of the mixture. (Assume no change in volume).

## Mixtures

As seen in the previous section, a mixture is a combination of two or more substances. Businesspeople mix different commodities before selling them in order to minimise on costs while maximising on profits. For example, in the market you may find one type of bean mixed with another.

***Example 14.5***

*Joseph has 24 kg of grade I rice which he sells at K 135 per kg. How many kilograms of grade II rice selling at K 110 per kg does he need to mix with in order to sell the mixed rice at K 118 per kg?*

***Solution***

*Total price for grade I = K (24 × 135) = K 3 240*

*Let the number of kilograms of grade II rice needed be y.*

*Total price for grade II = K 110y*

*Total price for the mixture*

*= price of 1 kg of mixture × number of kg*

$= 118(24 + y)$

$= 2\,832 + 118y$

*Total price for the mixture = total price for grade I + total price for grade II*

$= 3\,240 + 110y$

*Therefore,* $2\,832 + 118y = 3\,240 + 110y$

$$118y - 110y = 3\,240 - 2\,832$$

$$8y = 608$$

$$y = 51$$

*Thus, Joseph needs 51 kg of grade II rice.*

## Exercise 14.4

1. Angie has 48 kg of sugar costing K 160 per kg. How many kilograms of sugar costing K 200 per kilogram will she add in order to have a mixture costing K 176 per kg?
2. Patel mixed 80 kg of rice which was selling at K 180 per kilogram with another type which was selling at K 135. He had to sell a kilogram of the mixed rice at K 153 per kilogram. Find the mass of the second type of rice that was used in the mixture.
3. A trader mixed 990 kg of wheat costing K 150 per kg with another type of wheat costing K 164 per kg. If he sold the mixture at K 153 per kg, how many kilograms of the second type of wheat did he use in the mixture?
4. A businesslady mixed two types of wheat flour in order to sell the mixture at K 143 per kg. She mixed 300 kg of the first type costing K 140 per kg and 180 kg of the second type. Find the price of 1 kg of the second type.
5. A trader of food crops mixed 200 kg of high quality peas and 300 kg of low quality peas, which he was selling at K 72 per kg. Find the price of 1 kg of the high quality peas if he was selling 1 kg of the mixture at K

84.

6. Nthala bought 225 kg of sorghum flour at K 35 per kg. He then bought 150 kg of millet at K 40 per kg. He mixed the two types of flour in one sack. Find the average price per kg of the mixture of flour.
7. A lady mixed 500 kg of maize flour whose price is K 90 per kilogram with 300 kg whose price is K 70 per kg. What is the average price of 1 kg of the mixture?
8. A trader has 12 bottles of ghee whose price is K 300 per bottle and 8 other bottles of ghee whose price is K 240 per bottle. Find the average price of 1 bottle of the mixture.

# 15 VECTORS I

## Vector and scalar quantities

A group of rangers is preparing to go on a treasure hunt trip from their camping base. Before they leave the camp, they **must** know

1. the specific distance to the treasure, and
2. the direction in which they have to move.

If they do not know **both** the **distance** and the **direction** , they are not likely to locate the exact position of the treasure.

Quantities which have both magnitude (size) and direction are called **vector quantities** or simply **vectors** . Velocity and acceleration are examples of vectors.
Quantities which have no direction e.g. a packet of biscuits, the cost of a pen etc., are called **scalar quantities** or simply **scalars** .

The direction of a vector, in a diagram, is shown by means of an arrow.

***Example 15.1***

*Represent the vector 30 km/h due west by means of a diagram.*

***Solution***

*Fig. 15.1 shows the required representation.*


*Fig. 15.1*

Speed in a specific direction is known as **velocity** .

## Exercise 15.1

1. State whether the following are vector or scalar quantities.
   (a) A speed of 70 km/h due west
   (b) A distance of 50 km due east
   (c) 30 cows
   (d) 15 km
   (e) A speed of 400 km/h
   (f) A distance of 10 km south east of the city centre
   (g) 86 litres of milk
   (h) A distance of 70 km
   (i) A speed of 370 km/h on a bearing of 050°
2. Represent each of the vectors in Question 1 by an accurate well labelled diagram.
3. Name two examples, other than those in question 1, of
   (a) vector quantities,
   (b) scalar quantities.
4. Write down the magnitude and direction of each of the vectors in Fig. 15.2 .

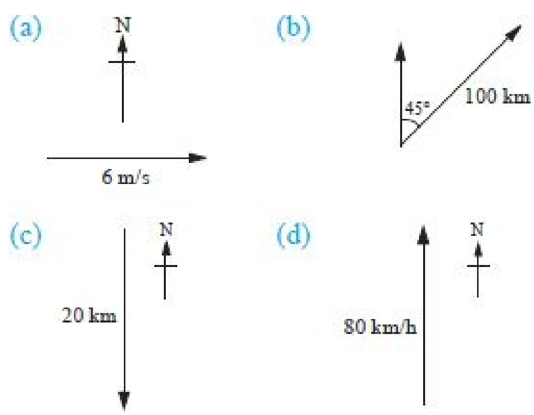



*Fig. 15.2*

## Displacement vector and notation

We have seen that a **directed distance** is a **vector** . To travel from town A to town B, along the shortest distance, we must travel in a specific direction and for a definite distance (Fig. 15.3 ).

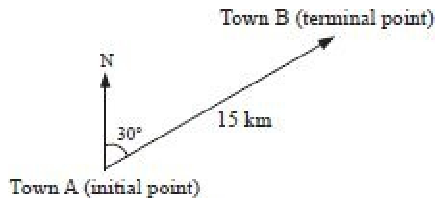



*Fig. 15.3*

The distance from A to B in the given direction is called the **displacement vector AB** , denoted as **AB** . Sometimes, vectors are denoted by specified small letters e.g. vector **a** .

In print vector **AB** is shown in bold.

In our handwriting we use an **arrow** or a **wavy line** notation since we cannot write in bold. For example, vector **AB** is written as $\overrightarrow{AB}$ or $\underset{\sim}{AB}$ ; vector **a** is written as $\underset{\sim}{a}$ .

A vector whose initial and terminal points coincide is a **null vector** denoted as **0** . Its magnitude is 0 (zero).

## Equivalent vectors

If two vectors have the same magnitude and the same direction, then they are **equivalent vectors** .

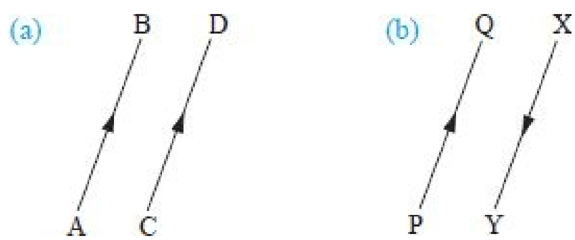



*Fig. 15.4*

(i) **AB** and **CD** (Fig. 15.4 (a)) are parallel. They have the same sense of direction, the same magnitude or length, denoted as **|AB|** and **|CD|** respectively. **AB** is therefore equivalent to **CD** . We write, **AB** = **CD** (sometimes written as **AB ≡ CD)** .

(ii) **PQ** and **XY** (Fig. 15.4 (b)) are parallel. They also have the same magnitude, but are in opposite directions. Therefore, **PQ** and **XY** are **not equivalent** . We can write **PQ** ≠ **XY** . We have to change the direction of either **PQ** or **XY** so that their sense of direction is the same. A negative sign is used to reverse the direction. Thus, **PQ** = **-XY**

.

In general, two vectors are equivalent if they have the **same magnitude** and the **same sense of direction** .

## Addition of vectors

Triangle ABC (Fig. 15.5 ) represents routes joining three towns A, B and C.

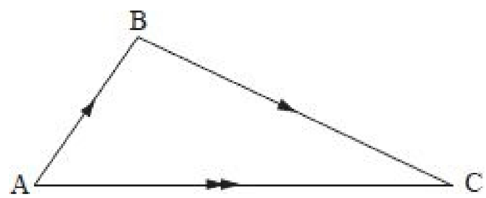


*Fig. 15.5*

If you go from A to B, then from B to C, the effect is the same as going from A to C directly.

The required effect is to reach town C from A.

Since the effect is the same, then

**AB** + **BC** = **AC** .

Vector **AC** is called the **resultant vector** of **AB** and **BC** . Such a vector is usually represented by a line segment with a double arrowhead, as in Fig. 15.5 .

***Example 15.2***

*Using Fig. 15.6 , write down the single vector equivalent to*

*(a)* ***AB*** *+* ***BC***

*(b)* ***AE*** *+* ***ED***

*(c)* ***BC*** *+* ***CD*** *+* ***DE***

*(d)* ***ED*** *+* ***DC*** *+* ***CB***

*(e)* ***AB*** *+* ***BA***

*(f)* ***CD + DC***

*(g)* ***AE + EB + BC***

*(h)* ***CD + DE + EB***

*(i)* ***AB + BC + CD + DE***

*(j)* ***DE + EA + AB + BC + CD***

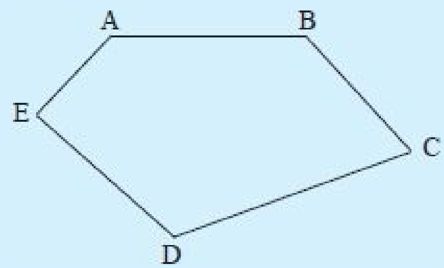



*Fig. 15.6*

## *Solution*

*(a)* ***AB + BC = AC*** *(Moving from A to B, then from B to C is equivalent to moving from A to C directly.)*

*(b)* ***AE + ED*** *= A to E then to D.*
*= A to D*
*=* ***AD***

*(c)* ***BC + CD + DE = BD + DE = BE***

*(d)* ***ED + DC + CB = EC + CB = EB***

*(e)* ***AB + BA = AB – AB = 0*** *(from A to B then back to A)*

*(f)* ***CD + DC = 0***

*(g)* ***AE + EB + BC = AB + BC = AC***

*(h)* ***CD + DE + EB = CE + EB = CB***

*(i)* ***AB + BC + CD + DE = AC + CD + DE***

$$= \boldsymbol{AD} + \boldsymbol{DE} = \boldsymbol{AE}$$

*(j)* $\boldsymbol{DE} + \boldsymbol{EA} + \boldsymbol{AB} + \boldsymbol{BC} + \boldsymbol{CD}$

$= \boldsymbol{DA} + \boldsymbol{AB} + \boldsymbol{BC} + \boldsymbol{CD}$

$= \boldsymbol{DB} + \boldsymbol{BC} + \boldsymbol{CD}$

$= \boldsymbol{DC} + \boldsymbol{CD}$

$= \boldsymbol{0}$ *(Start from D and back to D)*

## Subtraction of vectors

In Form 1, we learnt that integers such as 4 – 3 can be expressed as $4 + {}^{-}3$. Similarly, in subtraction of vectors, we can represent **a** – **b** as follows:

(i) Choose any vector **a** (Fig. 15.7 ).

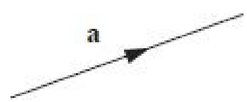



*Fig. 15.7*

(ii) Choose the particular vector $^{-}$**b** which starts at the end of **a** (Fig. 15.8 ).

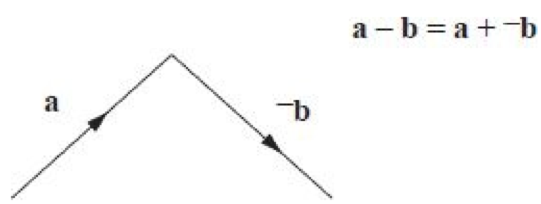



*Fig. 15.8*

(iii) Join the start of **a** to the end of **b** (Fig. 15.9 ).

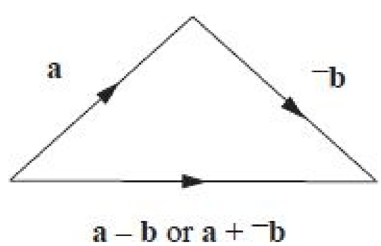



*Fig. 15.9*

## Exercise 15.2

1. Which of the vectors in Fig. 15.10 are equivalent?

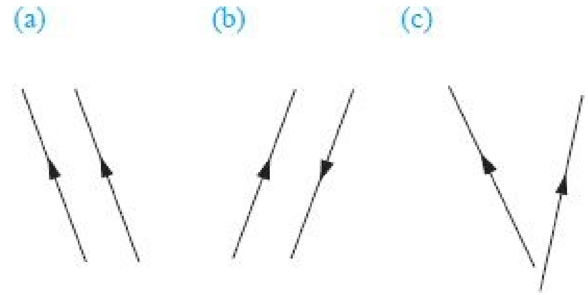



*Fig. 15.10*

2. STUR is a quadrilateral (Fig. 15.11 ). Use it to write down the single vector equivalent to
   (a) **ST** + **TU**
   (b) **TS** + **SR**
   (c) **RS** + **ST**
   (d) **UR** + **RS**
   (e) **UT** + **TR**
   (f) **UR** + **RT**
   (g) **TS** + **ST**
   (h) **UR** + **RU**
   (i) **RS** + **ST** + **TU**
   (j) **UT** + **TS** + **SR**
   (k) **ST** + **TU** + **UR** + **RS**
   (l) **UT** + **TS** + **SR** + **RU**

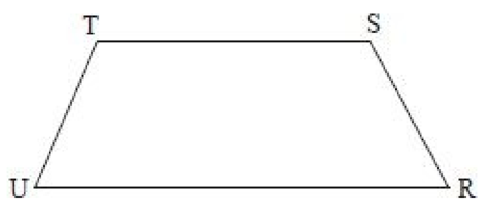



*Fig. 15.11*

3. Draw a triangle STR and put arrows on its sides to show **TS + SR = TR.**

4. Draw a quadrilateral ABCD and on it show **BC** , **CD** and **DA** . State a single vector equivalent to **BC + CD + DA** .

5. A man walks 10 km in the NE direction, and then 4 km due north. Using an appropriate scale, draw a vector diagram showing the man's

displacement from his starting point. When he stops walking, how far from the starting point will he have walked?

6. Vectors **a** , **b** and **c** are such that **a** = **b** and **b** = **c** . What can you say about **a** and **c** ?

7. Mr. Tsalani's family planned a sight seeing trip which was to take them from Lilongwe to Dedza, then to Mtakataka and back to Lilongwe. Draw a vector triangle to show their trip.

   What vector does **LM** + **DM** + **ML** represent if L stands for Lilongwe, D for Dedza and M for Mtakataka?

8. PQRS (Fig. 15.12 ) represents a parallelogram.

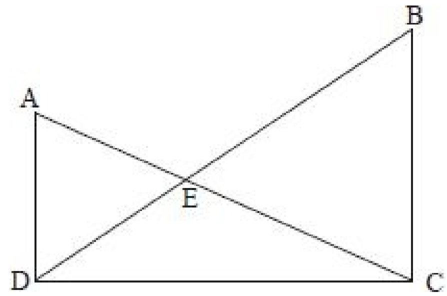
   (a) Copy the figure. Mark with arrows and name two pairs of equal vectors.

   (b) Simplify

   (i) **PQ** + **QR**
   (ii) **PS** + **SR**
   (iii) **SP** + **PQ**
   (iv) **SR** + **RQ**


9. For each of the following equations, use Fig. 15.13 to find a directed line segment which can replace **PQ** .

   (a) **AE** + **PQ** = **AB**
   (b) **DE** + **PQ** = **DB**
   (c) **DB** + **PQ** = **0**
   (d) **EB** + **PQ** = **EC**
   (e) **EB** + **PQ** = **ED**
   (f) **PQ** + **DA** = **CA**
   (g) **AE** + **ED** + **PQ** = **AD**
   (h) **AD** + **PQ** + **EC** = **AC**
   (i) **DC** + **PQ** + **ED** = **0**




10. Use Fig. 15.14 to simplify the following.

(a) **a – w**
(b) **u + a**
(c) $^{-}$ **w + u**
(d) **u + v**
(e) **u – b**
(f) **b – u**
(g) **v + w – a**

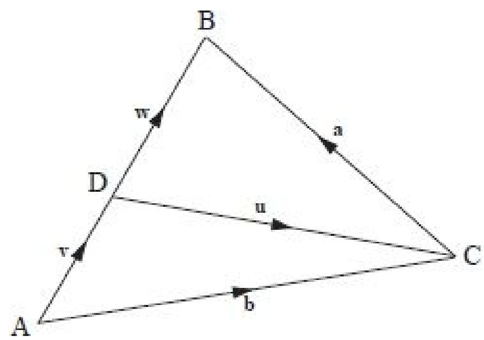
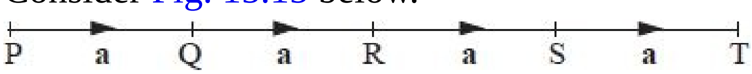



*Fig. 15.14*

## Scalar multiplication

Consider Fig. 15.15 below.




**PT** = **PQ** + **QR** + **RS** + **ST**

= **a** + **a** + **a** + **a**

= 4 × **a**

= 4**a**

This means that the length (magnitude) of **PT** is four times that of **a** .

Note that 4**a** and **a** have the same direction.

Similarly, **PR** = 2**a** and

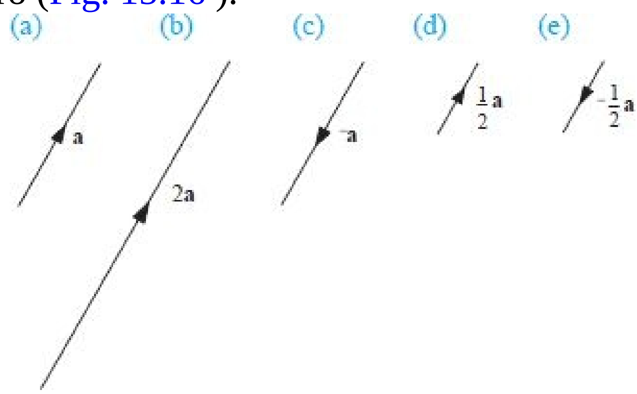
**PS** = 3**a** .

If X is a point halfway between P and Q, **PX** would be half of **a** .

i.e., **PX** = **a** .

In vectors **PT** $\frac{1}{2}$ , **PS** , **PR** and **PX** , the values 4, 3, 2 and are scalars.

A scalar $\frac{1}{2}$ multiplier can take any value, positive or negative, whole number or fraction, or even zero (Fig. 15.16 ).




In Fig. 15.16 parts (b) to (e), vector **a** has been multiplied by 2, $^-1$, $\frac{1}{2}$ and $^-\frac{1}{2}$ respectively.

> If is a scalar, and **a** is a vector, then **a** is a vector parallel to **a** , and times its length.
>
> If > 0, then **a** has the same sense of direction as **a** , and its magnitude is times that of **a** .
>
> If $m$ = 0, then $m$***a*** is a null vector whose magnitude is zero.
>
> If $m$ < 0, then $m$***a*** is a vector whose direction is opposite that of **a** and whose length is $|m|$ times that of **a** , where $|m|$, read as the **modulus of *m*** , means the value of $m$ irrespective of the sign.
>
> Multiplying a vector by a negative number reverses the direction of the vector.

## *Example 15.3*

*Towns A, B and C are along a straight road which runs due north from A. From A to B is 6 km and from B to C is 12 km.*

*Express in terms of **AB.***

*(a) **BC***

*(b)* ***AC***

*(c)* ***CB***

***Solution***

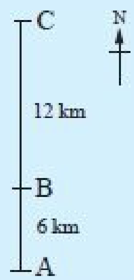



*Fig. 15.17*

*(a)* ***AB*** *is 6 km due north.*

***BC*** *is 12 km due north.*

***BC*** *= twice* ***AB***

*= 2****AB*** *(see Fig. 15.17 )*

*(b)* ***AC = AB + BC*** *due north*

*= 6 + 12 km due north*

*= 18 km due north*

*∴* ***AC*** *= 3****AB***

*(c)* ***CB*** *is 12 km due south.* ***CB*** *has same magnitude as* ***BC*** *but is in the opposite direction.*

*∴* ***CB*** *= ⁻* ***BC***

*= ⁻ 2****AB***

## Exercise 15.3

1. Represent each of the following vectors by a diagram.

   (a) A speed of 400 km/h due west.

   (b) A speed of 60 km/h on a straight road due east increased in the ratio 5 : 4.

   (c) Half the reverse of a speed of 80 km/h due north on a straight road.

2. Four railway stations L, M, E and R are on a straight railway line (Fig. 15.18 ).

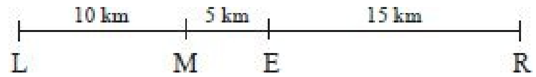



*Fig. 15.18*

(a) Express the following in terms of **ME.**

(i) **LM**
(ii) **LE**
(iii) **ER**
(iv) 2**ER**
(v) **MR**
(vi) 3**LM**

(b) Express the following in terms of **LM** .

(i) **MR**
(ii) **LE**
(iii) **ML**
(iv) **ME**
(v) $\frac{1}{2}$ **ME**
(vi) **LR**
(vii) **RM**
(viii) $\frac{1}{3}$ **ER**

(c) Express the following in terms of **RE.**

(i) **EM**
(ii) **ME**
(iii) **LE**
(iv) **RL**
(v) **ML**
(vi) **LM**
(vii) **EL**
(viii) **LR**

(d) Write the following in terms of **ME.**

(i) **LM + EM**
(ii) **LM + RE**
(iii) **ER + ML**
(iv) **ME + RE**

(v) **ME + ML**
(vi) **ER +** $^{-}$ 3**ME**
(vii) **LM – ER**
(viii) **LM + ME + ER**

## Vectors in the Cartesian plane

In Fig. 15.19 , **AB** represents a vector, namely a translation of the points on the plane, which moves A to B, H to K, P to Q, etc. This translation transfers each and every point on the plane 1 unit to the right and 3 units up. We can indicate this vector by an ordered pair of numbers which we write in column form as $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ . This is called a **column vector** for the said translation. 1 and 3 are called **components** of the vector.

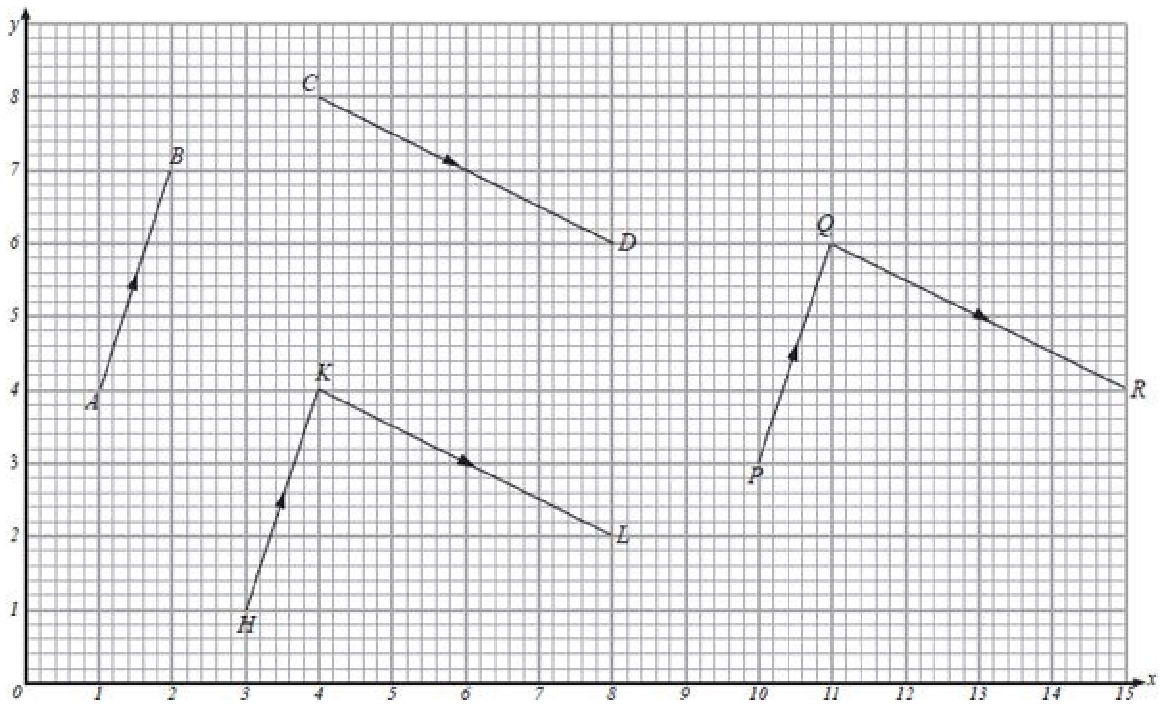



*Fig. 15.19*

### *Example 15.4*

*Using Fig. 15.19 (previous page), find the column vectors which represent*

*(a)* *(i)* ***AB***
*(ii)* ***CD***

*(iii)* ***BA***
*(iv)* ***DC***
*(v)* ***HL***
*(vi)* ***PR***
*(vii)* ***LH***

*(b) How can we obtain the column vectors for* ***HL*** *and* ***PR*** *from those of*
*(i)* ***HK*** *and* ***KL***
*(ii)* ***PQ*** *and* ***QR*** *respectively?*

## Solution

*(a) (i) Point A moves 1 unit to the right and 3 units up to map onto B.*

$\therefore \mathbf{AB} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

*(ii) Point C moves 4 units to the right and 2 units down to map onto D.*

$\therefore \mathbf{CD} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

*(iii) B moves 1 unit to the left and 3 units down to A.*

$\therefore \mathbf{BA} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$

*(iv) D moves 4 units to the left and 2 units up to C.*

$\therefore \mathbf{DC} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$

*(v) H moves 5 units to the right and 1 unit up to L.*

$\therefore \mathbf{HL} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

*(vi) P moves 5 units to the right and 1 unit up to R.*

$\therefore \mathbf{PR} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

*(vii) L moves 5 units to the left and 1 unit down to H.*

$\therefore \mathbf{LH} = \begin{pmatrix} -5 \\ -1 \end{pmatrix}$

*(b) (i) Since* $\mathbf{HL} = \mathbf{HK} + \mathbf{KL}$ *,*

*then* $\mathbf{HL} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

*(We add corresponding components in the column vectors* ***HK*** *and* ***KL*** *)*

*(ii)* ***PR = PQ + QR***

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Note that given two vectors **u** and **v** such that $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} c \\ d \end{pmatrix}$ where *a*, *b*, *c* and *d* are scalars, $\mathbf{u} + \mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$. Also, $\mathbf{u} - \mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a - c \\ b - d \end{pmatrix}$.

## Exercise 15.4

Use Fig. 15.20 (next page) to answer Questions 1 to 3.

1. (a) Name all the vectors that are equal to **AB** and state their column vectors.
   (b) Name the vector which is equal to **EF** .
   (c) Is **PQ** equal to **KL** ? Give a reason for your answer.
   (d) Is **KL** equal to **QP** ? Why?
   (e) Simplify **EF + FG + GH** and give your answer as a column vector.
   (f) Name a resultant vector which is equal to **NM** .
   (g) Name three vectors which are equal to 2**GL.**
   (h) Name a vector which is parallel to **GH** .
2. Write all the vectors in Fig. 15.20 as column vectors.
3. Simplify **FG + GH** giving your answer in column vector form. What is the meaning of the first component in your answer?
4. Draw diagrams on squared paper to show
   (a) $\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$
   (b) $\begin{pmatrix} 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$
   (c) $\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$
5. Simplify
   (a) $\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

(b) $\begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} ^-2 \\ 4 \end{pmatrix} + \begin{pmatrix} ^-3 \\ ^-8 \end{pmatrix}$

(d) $\begin{pmatrix} ^-10 \\ ^-4 \end{pmatrix} + \begin{pmatrix} ^-6 \\ 8 \end{pmatrix}$

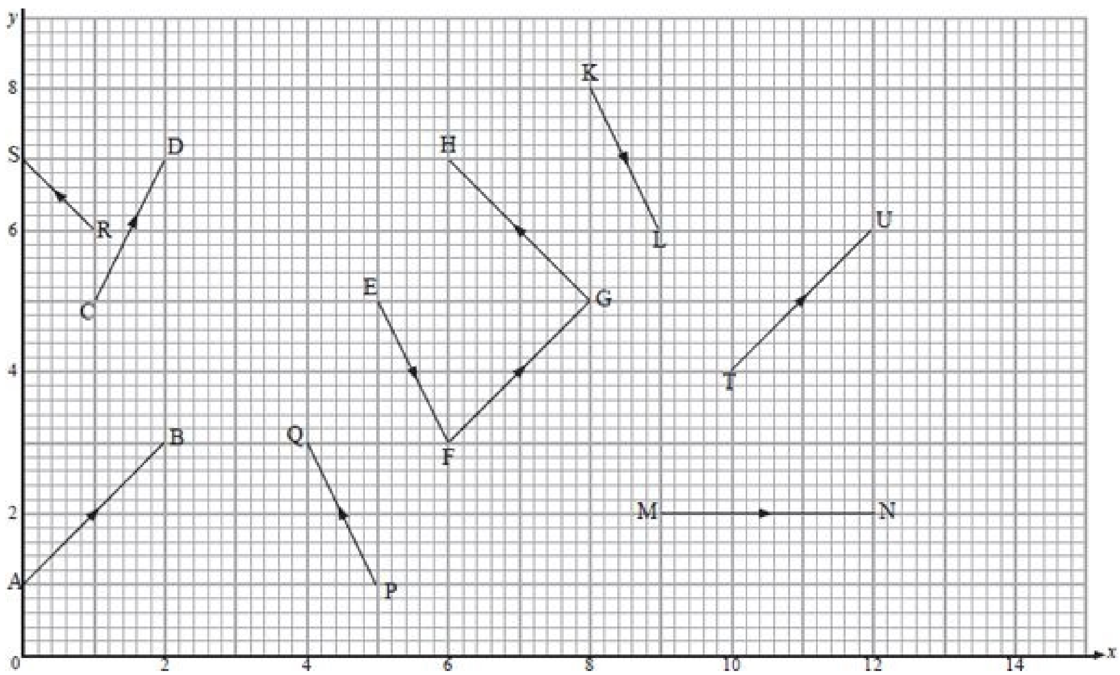



*Fig. 15.20*

6. Given that 2**u** means **u** followed by **u** , $2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ means $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ followed by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ which means a total of 2 units to the right and 4 units up i.e. $2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \times 1 \\ 2 \times 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

Simplify

(a) $3\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

(b) $2\begin{pmatrix} 5 \\ 6 \end{pmatrix}$

(c) $3\begin{pmatrix} ^-1 \\ 5 \end{pmatrix}$

(d) $\frac{1}{3}\begin{pmatrix} ^-9 \\ 0 \end{pmatrix}$

(e) $-\frac{1}{2}\begin{pmatrix} ^-6 \\ 4 \end{pmatrix}$

(f) $^-3\begin{pmatrix} 2 \\ ^-1 \end{pmatrix}$

(g) $4\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

(h) $4\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

7. If $\mathbf{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, evaluate

(a) $\mathbf{a} + \mathbf{b}$
(b) $2\mathbf{a}$
(c) $3\mathbf{b}$
(d) $\mathbf{b} + \mathbf{c}$
(e) $4\mathbf{a} + 3\mathbf{b}$
(f) $^{-}\mathbf{a} + 2\mathbf{c}$
(g) $3(\mathbf{a} + \mathbf{b})$
(h) $\mathbf{a} - \mathbf{b}$
(i) $\mathbf{a} + \mathbf{c}$
(j) $\mathbf{a} + \mathbf{b} + \mathbf{c}$
(k) $2\mathbf{b} + 5\mathbf{c}$
(l) $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$
(m) $^{-}2(\mathbf{a} + \mathbf{c})$
(n) $\mathbf{a} - \mathbf{c}$

## Translation and displacement vectors Translation

We are going to consider a displacement process under which each point of a plane figure is moved onto another point, a certain distance away, in a certain direction (Fig. 15.21 ).

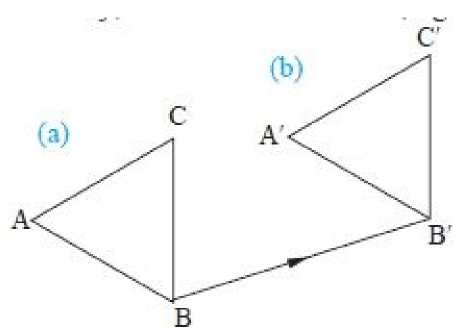



*Fig. 15.21*

### *Activity 15.1*

1. Copy triangle ABC (Fig. 15.21 (a)) on a tracing paper.

2. Draw the line segment joining B to B′ as shown in Fig. 15.21 .
3. Slide the tracing using line BB′ as a guide line, to ensure that B moves onto B′ in a straight line.
4. When B coincides with B′ stop the slide. What do you notice about the positions of A and C?
5. What do you notice about the new position of ΔABC? What can you say about the two triangles?
6. Using a ruler, measure the lengths of AA′, BB′ and CC′. What do you notice about the lengths? Are the lines parallel?

We notice that each point on triangle ABC has moved the same distance in the same direction. The process that moves triangle ABC onto triangle A′B′C′ is called a **translation** .

In general, under a translation:

1. All the points on the object move the same distance.
2. All the points move in the same direction.
3. The object and the image are identical and they face the same direction. Hence, they are **directly congruent** .
4. A translation is fully defined by stating the **distance and direction** that each point moves.

## Translation in the Cartesian plane

When using the Cartesian plane, a translation is fully defined by stating the distances moved in the x and y directions. The column vector that defines the translation is also called the **displacement vector** of the translation.

### *Example 15.5*

*Triangle ABC has vertices A (0, 0), B (5, 1) and C (1, 3). Find the coordinates of the points A ', B ' and C ', the images of A, B and C respectively, under a translation with displacement vector* $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ .

***Solution***

*A displacement vector* $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ *means that each point on ABC moves 2 units in the positive direction of the x-axis followed by 5 units in the positive direction of the y-axis (Fig. 15.22 ).*

*Thus* $A(0, 0) \rightarrow A'(0 + 2, 0 + 5)$, *i.e.* $A'(2, 5)$

$B(5, 1) \rightarrow B'(5 + 2, 1 + 5)$, *i.e.* $B'(7, 6)$

$C(1, 3) \rightarrow C'(1 + 2, 3 + 5)$, *i.e.* $C'(3, 8)$

$\therefore \Delta A'B'C'$ *is the image of* $\Delta ABC$ *under translation, displacement vector* $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

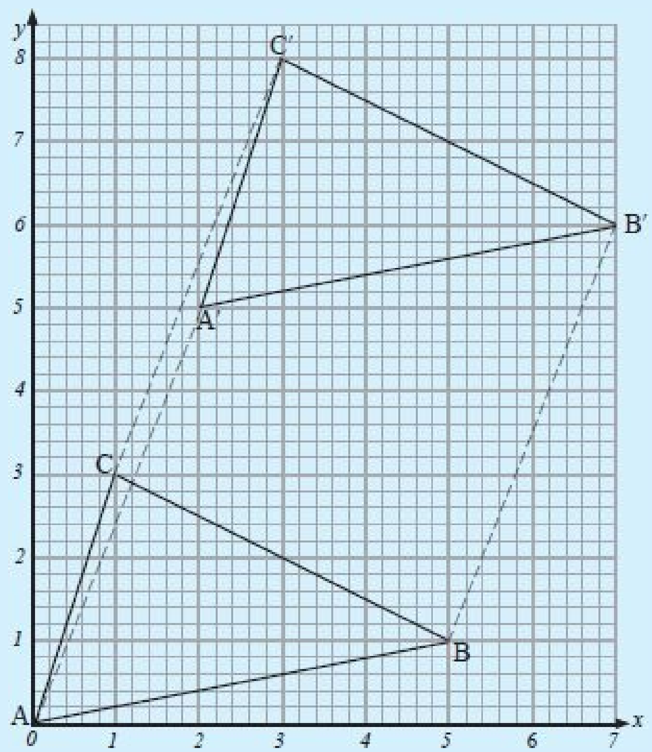



*Fig. 15.22*

## Exercise 15.5

1. A packaging case is pushed (without turning) a distance of 8 m, in a straight line (Fig. 15.23 ). The corner of the box which was originally at A ends up at B.

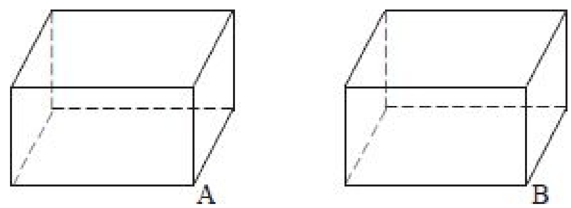



*Fig. 15.23*

How far will each of the following have moved?

(a) The upper front edge
(b) Each vertex
(c) The centre of each face
(d) The centre of the box

Fig. 15.24 shows a tiling composed of congruent parallelograms of sides 10 cm by 5 cm. The lines in the diagram are used as guidelines along which tiles may be slid. Use the figure to answer Questions 2 to 5.

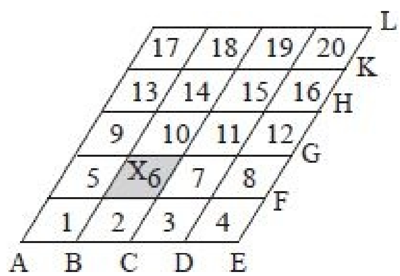



*Fig. 15.24*

2. (a) If tile 6 moves onto tile 7, in what direction has each vertex of the tile moved? How far has the point X moved? Make a statement that is true for every point on tile 6.

   (b) Tile 6 moves along the guideline parallel to line BC in the direction BC. Observe and state the direction of motion of

      (i) each vertex of tile 6.
      (ii) the point X on tile 6.
      (iii) every point on tile 6.

   (c) Answer Question (b) for the case where tile 6 is slid to position 14.

3. Name the tiles onto which tiles 1, 11, 15 and 18 will be translated by a translation equivalent to that of Question 2 (a).

4. (a) Name the tiles onto which tiles 2, 8, 11 and 5 will be translated by a

translation equivalent to that of Question 2 (c).

(b) What will be the images of letters E, F, G and H under the same translation?

5. Write down all the possible translations that are equal to the translation

(a) **FH**

(b) **AC**

6. State which of the following statements are true, and which are false.

When a figure or an object has a translation applied to it,

(a) all points move in the same direction.

(b) not all points of the figure move in the same direction.

(c) all lengths in the object remain unchanged.

(d) usually, at least one point on the figure remains unchanged.

(e) a translation can be described by many directed line segments, provided each has the same length and same direction.

7. Find the image of ΔABC, where A is ($^-3$, $^-2$), B is ($^-1$, 1) and C is (2, $^-1$), under translation vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

8. ΔABC with A (0, 1), B (2, 0) and C (3, 4) is given a translation equivalent to $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$. followed by $\begin{pmatrix} ^-3 \\ 2 \end{pmatrix}$.

Find the coordinates of the final image.

9. Given that **a** is a vector, solve the equation

(a) $2\mathbf{a} + \begin{pmatrix} 4 \\ ^-1 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \end{pmatrix} - \mathbf{a}$

(b) $4\mathbf{a} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} ^-2 \\ 4 \end{pmatrix} = \mathbf{0}$

# 16 GEOMETRIC CONSTRUCTIONS II

## Definition of locus

Suppose a particle P moves in a plane so that its distance from a fixed point O is always the same, say 2 cm. In its motion, P obeys the condition or law that ‘P is always 2 cm away from the fixed point O’. The motion of P describes a path which forms the circumference of a circle centre O, radius 2 cm. Such a path, as the one traced out by point P, is called the **locus** of P.

Literally, the word ‘locus’ means **location** , **position** or **place** . However, the above description of the motion of particle P demonstrates the simple definition of locus, which is: **the path traced out by a moving point** . Nevertheless, a more comprehensive definition of locus is as follows:

> A **locus** (plural: **loci** ) of a point is the set of all possible positions occupied by a point which varies its position according to some given law.

The expressions ‘locus of a point’ and ‘locus of points’ are often used interchangeably. Whichever we use depends on how we think of the problem. If, for example, we are thinking about a point moving under stated restrictions, ‘locus of a point’ might sound better. If we think of the problem as a series of points which satisfy given conditions, then ‘locus of points’ may sound better.

## Common types of loci

The following activities are intended to help us define and illustrate some common loci. We will use both geometrical and/or graphical approach, whichever is appropriate.

## Constant radius locus

## (Locus of a point at a fixed distance from a fixed point)

***Activity 16.1***

1. On a sheet of paper, mark a point O.
2. Mark at least 10 more points on your sheet such that each point is 3 cm from O.
3. How many more such points can you mark on your figure?
4. Draw a smooth curve through the points in (2) to show all the possible points that are 3 cm away from O.
5. Describe fully the series of points in (4) above.
6. If O is a point in space, what will be the locus of P?

From Activity 16.1 , we see that:

In two dimensions, the locus of points *d* cm from a fixed point, O, is a circle centre O, radius *d* cm. In three dimensions, the locus is a spherical shell centre O, radius *d* cm.

## Constant distance from a fixed line

## (Locus of points at a fixed distance from a given straight line)

### *Activity 16.2*

1. In your exercise book draw a line *l* .
2. Mark several points on either side of *l* , a distance of 2 cm from *l* . Is it possible to locate some more points which satisfy the same condition?
3. Describe the locus of all points that are 2 cm from *l* .
4. If *l* is a line in space, what will be the locus of P?

From Activity 16.2 , we see that:

In two dimensions, the locus of points that are *d* cm away from a given line *l* is a pair of lines, on either side of *l* , each *d* cm from *l* and therefore parallel to *l* [ Fig. 16.1 (a) ].

In three dimensions, the locus is a cylindrical curved surface of radius *d* cm, with *l* as the central axis. [ Fig. 16.1 (b) ].

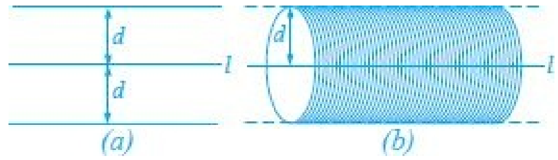



*Fig. 16.1*

## The perpendicular bisector locus

## (Locus of a point equidistant from two given points)

### *Activity 16.3*

1. Draw a line segment AB, 7 cm long.
2. Mark a point P on AB such that AP = PB (Fig. 16.2 ).

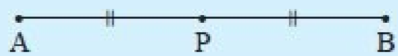



*Fig. 16.2*

3. Mark other points not on the line, that are equidistant from A and B. Describe the locus of these points.
4. If AB is a line segment in space what will be the locus of P?

From Activity 16.3 , it is evident that:

The locus of points equidistant from two given points A and B is the perpendicular bisector of the line segment joining the two points [ Fig. 16.3 (a) ]. In three dimensions, the locus is a plane surface which bisects line segment AB perpendicularly [ Fig. 16.3 (b) ].

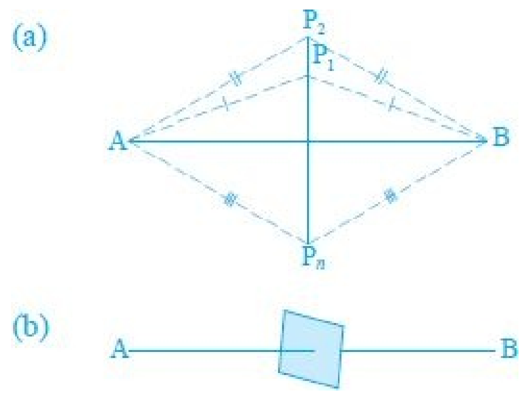



*Fig. 16.3*

## Angle bisector locus

## (Locus of a point equidistant from two intersecting lines)

***Activity 16.4***

1. Draw an acute angle AOB (Fig. 16.4 ).

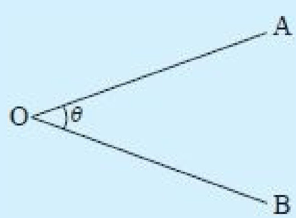



*Fig. 16.4*

2. Locate a series of points that are halfway between OA and OB (Fig. 16.5 ). Hence indicate all the points equidistant from OA and OB.

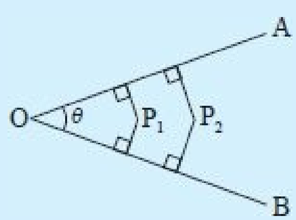



*Fig. 16.5*

3. Describe the locus of the points in (2) above.

Activity 16.4 shows that:

Given any angle AOB, the locus of points that are equidistant from OA and OB is the bisector of angle AOB.

## Constant angle locus

Given two fixed points A and B, we can find the locus of a point P, which moves such that ∠APB is always a constant.

## 90 ° angle locus

### *Activity 16.5*

1. Draw a line segment AB, 10 cm long.
2. At point A, draw a series of straight lines at 10° intervals (Fig. 16.6 ), taking line AB to be at 0°.

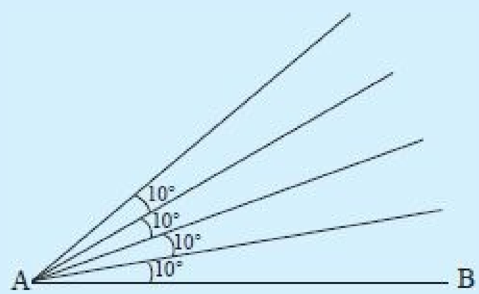



*Fig. 16.6*

3. From B construct a perpendicular to each of the lines drawn through A, and label the points where the perpendiculars meet the lines as $P_1$ , $P_2$ … (Fig. 16.7 ).

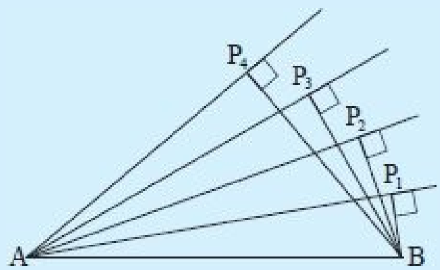


*Fig. 16.7*

4. Using a smooth curve, join the series of the points P.
5. Describe the locus of P.
6. On the same Fig. 16.7 , locate another possible series of points P, which satisfy the condition in 3 above.
   Describe the complete locus of P.

Observation:

The locus of the point P is the circumference of a circle with AB as its diameter. This locus illustrates that the angle subtended by a diameter on a semi-circle is always equal to 90°.

## Exercise 16.1

1. Given two parallel lines *l* and *m*, describe the locus of points equidistant from *l* and *m*. Sketch the locus.
2. Given lines MN and AB which intersect at a point O, sketch and describe the locus of points equidistant from the two lines.
3. Describe, the locus of
   (a) the tip of the minute hand of a clock in a time of 20 minutes.
   (b) the pendulum bob, P, of a clock as the pendulum swings to and fro.
   (c) the centre of a round coin as the coin rolls along a straight line on a table.
   (d) the key hole in a door as the door opens through 90° from its shut position.
4. On the rim of a round coin, there is a small dent marked A. The coin rolls along a straight line on a table. If the coin starts with the dent on the line, sketch the locus of A as the coin rolls.
5. AB is a fixed line, 15 cm long. A particle P moves in the plane of AB so that it is always 2 cm from AB. Sketch the locus of P.
6. Lay a sheet of paper on a drawing board and hold it down with two drawing pins 4 cm apart. Make a loop of thread 10 cm long. Place the

loop over the pins (Fig. 16.8 ). Draw the thread taut with the tip of a pencil. Keeping the thread taut, move the pencil so that the tip traces its locus on the paper. Sketch the locus of the tip of the pencil.

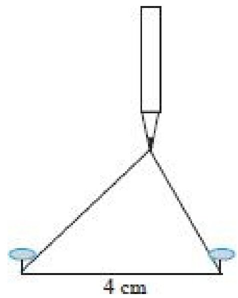



*Fig. 16.8*

7. In each of the following cases, describe the required locus and illustrate your answer using a sketch diagram.
   (a) The locus of points equidistant from P(0, 0) and Q(2, 0).
   (b) The locus of points equidistant from A($^-$1, $^-$1) and B(1, 1).
   (c) The locus of points equidistant from points C(0, 0) and D(3, 3).
   (d) The locus of points equidistant from the line $y = x$ and $y = -x$ .
8. A point P moves so that it is always 5 cm from the origin. Describe the locus of P and give its equation in terms of $x$ and $y$ .
9. A and B are two fixed points 5 cm apart. Sketch and describe the locus of point P such that
   (a) ∠APB = 90°
   (b) ∠APB > 90°
   (c) ∠APB < 90°
10. ΔABC is one of a series of right-angled triangles constructed on either side of a given base AB as hypotenuse. Sketch and describe the locus of C.
11. ΔABP stands on a fixed base AB, and has a constant area. If ∠APB is constant, sketch the locus of P.

## Inscribed circles

An **inscribed circle** is a circle drawn inside a triangle such that the circle touches the three sides of the triangle (Fig. 16.9 ).

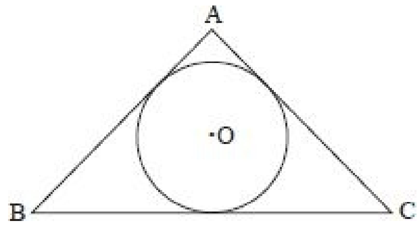



*Fig. 16.9*

The centre of an inscribed circle is called the **in-centre** and the circle is also called the **in-circle** .

The radius of the circle is called the **in-radius** .

## Constructing the inscribed circle of a given triangle

### *Activity 16.6*

1. Draw any triangle ABC.
2. Bisect any two angles, say ∠ABC and ∠ACB. The point of intersection O is the centre of the required circle.
3. From O, construct a perpendicular to BC, meeting BC at N. (The perpendicular from O could be drawn to any of the sides).
4. With centre O, radius ON, draw a circle.

This is the required circle (Fig. 16.10 ).

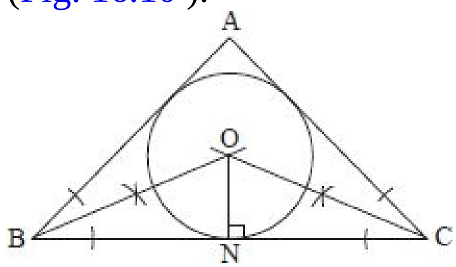



*Fig. 16.10*

Note that to locate the centre of the circle we use the fact that:

The internal bisectors of the three angles of a triangle are concurrent, i.e. they meet at a common point which is equidistant from the three sides of the triangle.

This is referred to as the **in-centre theorem** .

## Circumscribed circles

A circle which passes through all the three vertices of a triangle, as in Fig. 16.11 , is called the **circumscribed circle** or the **circumcircle** of the triangle. The centre of the circle is called the **circumcentre** . The radius of this circle is called the **circumradius** .

For any ΔABC, if O is the circumcentre (Fig. 16.11 ), then the circumradius = OA = OB = OC.

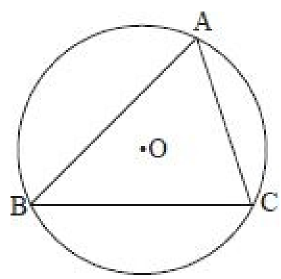



*Fig. 16.11*

## Constructing the circumcircle of a given triangle

### *Activity 16.7*

1. Draw any ΔABC.
2. Construct perpendicular bisectors of any two sides, say BC and AC, to meet at O.
3. With O as centre, radius OA or OB or OC, draw a circle.

The circle passes through all the three vertices A, B and C.

This is the circumcircle of ΔABC (Fig. 16.12 ).

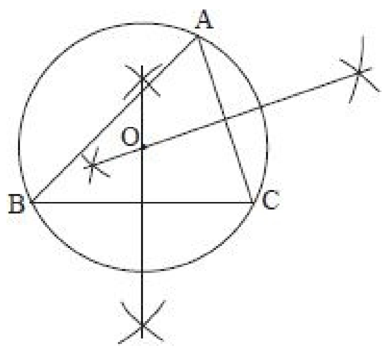



*Fig. 16.12*

Note that if the perpendicular bisector of AB was drawn, it would also pass through the circumcentre O.

The perpendicular bisectors of the three sides of a triangle are concurrent.

This is called the **circumcentre theorem** . It leads as to the following theorem.

There is one, and only one circle, which passes through three given points which are not on the same straight line.

***Example 16.1***

*Find the radius of the circle which circumscribes an isosceles triangle of sides 15 cm, 15 cm and 9 cm.*

***Solution***

*In Fig. 16.13 , O is the circumcentre and AD is the perpendicular from A to BC.*

$\therefore$ *AD bisects BC* $\Rightarrow$ *BD = 4.5 cm*

*Let the radius OB = r cm*

*In* $\Delta ABD$, $AD^2 = AB^2 - BD^2$

*(Pythagoras' theorem)*

$= 15^2 - 4.5^2$

$= 225 - 20.25$

$= 204.75$ *or* $204.8$

$\therefore AD = 14.31 \ cm$

$OD = AD - r = 14.31 - r$

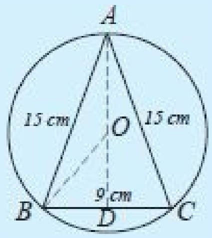



*Fig. 16.13*

In $\Delta OBD$, $OB^2 = OD^2 + BD^2$

i.e. $r^2 = (14.31 - r)^2 + 4.5^2$

$r^2 = 204.8 - 28.62r + r^2 + 20.25$

$\Rightarrow 28.62\ r = 225.05$

$r = \frac{225.05}{28.62}$

$= 7.863 \ cm$

$\approx 7.9 \ cm$ (2 s.f.)

## Exercise 16.2

1. An equilateral triangle of sides 3 cm is inscribed in a circle. Find the radius of the circle.
2. An equilateral triangle is inscribed in a circle of radius 4 cm. Find the length of the side of the triangle.
3. An isosceles triangle has sides 5 cm, 5 cm and 6 cm. Find the radius of its circumcircle.
4. Draw a triangle ABC such that AB = 6 cm, BC = 8 cm and $\angle ABC = 60°$. Construct the inscribed circle. Measure its radius.
5. Draw a triangle PQR such that PQ = 6.9 cm, QR = 5.4 cm and PR = 7.8 cm. Construct the inscribed circle. Measure its radius.
6. Draw a triangle with sides 8 cm, 8 cm and 5 cm and construct its inscribed circle. Measure the radius of this circle.
7. Draw a circle of radius 5 cm. Inscribe an equilateral triangle in the circle. Measure its side.

8. Draw a circle of radius 2.5 cm and circumscribe an equilateral triangle about the circle. Measure its sides.

# 13-16 REVISION EXERCISES 3

## Revision exercise 3.1

1. Given that **a** = $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ , **b** = $\begin{pmatrix} -5 \\ 4 \end{pmatrix}$ and **c** = $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ , find

   (a) **a** + **b**
   (b) **b** – **c**
   (c) **c** – **a**
   (d) **a** + **b** + **c**

2. P and Q are points whose coordinates are ($^-$2, 4) and ($x$ , $y$ ) respectively. B is another point (2, 0) such that **PQ** = 3**QB** . Find $x$ and $y$.

3. Given that A(2, 2), B(0, $^-$3) and C(2, 4), find the coordinates of D such that

   **AD** + 2**BC** = $\begin{pmatrix} 2 \\ 10 \end{pmatrix}$

4. Solve the following quadratic equations.

   (a) $(2x - 5)(x + 3) = 0$
   (b) $(x - 2)(3x + 4) = 0$
   (c) $(2x - 5)^2 = 0$
   (d) $(3x - 7)(x - 1) = 0$

5. Two consecutive odd numbers have a product 195. Find the numbers.

6. Solve the equation

   $$\frac{4}{x} - 1 = \frac{3 - x}{x + 3}$$

7. Use a ruler and a pair of compasses only to construct ΔABC such that AB = 6 cm, AC = 4 cm and ∠BAC = 30º. Inscribe a circle in ΔABC and calculate the area inside the triangle but outside the inscribed circle.

8. Draw a line segment AB = 6 cm. Construct

   (a) the locus of points equidistant from A and B,
   (b) the locus of points P such that the area of ΔAPB = 12 cm$^2$ ,
   (c) the locus of points C such that ΔABC is an isosceles triangle of

area 12 $cm^2$ .

9. Construct $\Delta PQR$ with PQ = 6 cm, QR = 4.4 cm and PR = 5.2 cm. On the same diagram, construct the locus of points equidistant from RP and QP.

10. Three types of coffee A, B and C are mixed in the ratio 2 : 3 : 5 by mass. Type A coffee costs K 210 per kg, type B K 160 per kg and type C K 120 per kg. The blend is then sold at a profit of 30%. Determine the selling price of the blend per kilogram.

11. On a new year's day, a lady prepared 12 litres of passion fruit juice for sale. She later bought 10 litres of ready-made juice from a supermarket at K 2 000. She then mixed the two types of juice and sold the mixture at K 250 per litre. What was the price per litre of the juice she had prepared?

12. A metal disc has a mass of 9.15 g. The disc has a diameter and a thickness of 0.148 cm. Calculate its density in grams per cubic centimetre.

## Revision exercise 3.2

1. OAB is a triangle. O is the origin, A is (6, 4) and B is (4, 8). Express **AB** as a column vector.

2. Given that the column vectors of **AB** = $\begin{pmatrix} -9 \\ -2 \end{pmatrix}$ **BC** = $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ and **CD** = $\begin{pmatrix} 10 \\ 5 \end{pmatrix}$ , evaluate

   (a) **AB** + **BC**
   (b) **AB** – **BC**
   (c) $\frac{1}{3}\mathbf{AB} - \frac{1}{3}\mathbf{CD}$

3. If $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, evaluate

   (a) **a** + **b**
   (b) 5**a**
   (c) 5**a** + 5**b**

4. The length of a hall is 3 m longer than its width. Find the length and width of the hall if its area is 270 $m^2$ .

5. Solve the quadratic equation.
   $x^2 + 2x - 80 = 0$
6. I think of number $x$, square it and subtract three times the original number. My answer is $^-2$. Find the number $x$.
7. Construct ΔABC with AB = 8 cm, BC = 6 cm and AC = 7cm. On the same diagram, construct the locus of P equidistant from B and C.
8. Using a ruler and a pair of compasses only,
   (a) construct ΔPQR in which QR = 5 cm, PR = 4 cm and ∠PRQ = 105º.
   (b) measure PQ.
   (c) construct and define the locus of all points which are equidistant from PR and QR.
9. Construct an equilateral triangle ABC. On it, construct the locus of a point such that ∠APC = 30º.
10. Ekari bought maize and beans at K 30 per kg and K 40 per kg respectively. She mixed them in a ratio such that after selling the mixture at K 35 per kg, she made a profit of 50%. Find the ratio in which she mixed them.
11. A mixture of beans is made from 26 kg of red coloured beans costing K 25 per kilogram and 104 kg of brown coloured beans. Find the price of 1 kg of brown coloured beans if 1 kg of the mixture is sold at K 21.
12. An alloy contains 30% by mass of lead (density = 11.4 g/cm$^3$) and 70% by volume of tin (density = 7.3 g/cm$^3$). Find the relative density of the alloy.

## Revision exercise 3.3

1. After a certain translation, point M($^-3$, $^-2$) is mapped onto point M′(5, $^-4$).
   (a) Find the translation vector.
   (b) Find the coordinates of point A, whose image is A′(3, $^-1$).
2. If (a) $\begin{pmatrix} -2 \\ 3 \end{pmatrix} + 3\mathbf{p} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$, find $\mathbf{p}$.

(b) $\begin{pmatrix} ^-8 \\ 2 \end{pmatrix}$ + **SR** = $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$, find **SR**.

3. Given that **a** and **b** are vectors, simplify the following:
   (a) 2**a** + **a**
   (b) 3**a** + 2(**a** + **b** )
   (c) 2(2**a** + **b** ) + 3(**a** + 2**b** )

4. Solve the following quadratic equations
   (a) $2x^2 - 3x = 0$
   (b) $2m^2 - 5m + 3 = 0$
   (c) $16x^2 + 9 = 24x$
   (d) $3x^3 - 12x = 0$

5. Find the values of $x$ which satisfy the equation.
   $$\frac{2x-1}{6} = \frac{x+1}{3x}$$

6. Given that $\frac{3}{m} - 4m = 2 - \frac{9}{m}$, find the value of $m$ .

7. Plot points A(1, 0) and B(4, 1). Using a ruler and a pair of compasses, construct the locus of a point O such that P is equidistant from A and B.

8. In a single diagram, using a pair of compasses and a ruler only, construct
   (a) ΔABC in which AB = 6 cm,
   BC = 8 cm and CA = 5 cm.
   (b) the locus of a point P, 3 cm from A and equidistant from B and C.
   (c) the locus of point Q which lies outside ΔABC, 3 cm from A such that ∠QAB = ∠ABC.

9. Brass is made by mixing zinc, tin and copper in the ratio 1 : 3 : 4 by mass. Copper costs K 122 000 per tonne, tin K 216 000 per tonne and zinc K 160 000 per tonne. Calculate the cost of making one kilogram of brass.

10. A metal A is an alloy of two metals B and C. Metal B has a mass of 68 g and a density of 17 g/cm$^3$ . Metal C has a mass of 18 g and a density of 3 g/cm$^3$ . Find the density of the alloy.

11. A rectangular tank is 2.1 m by 0.9 m by 1.2 m.

(a) Find the volume of the tank in $m^3$ .

(b) If the density of a liquid is 1.2 tonnes/$m^3$ , find the mass of the liquid needed to fill the tank.

12. (a) Draw rectangle STU such that ST = 3.6 cm, TU = 3.9 cm and SU = 2.7 cm.

    (b) On ΔSTU, construct the locus of point P such that ∠TPU is equal to ∠TSU and that P is always on the same side of TU as S.

# MODEL PAPER I

**Answer all the twenty questions (100 marks).**

1. Solve the equation $\frac{1}{-x-1} = \frac{1}{x+4}$

*(4 marks)*

2. Simplify the expression
   $5a - 4b - 2\,[a - (2b + c\,)]$.

*(4 marks)*

3. Solve the inequality $3 - 2x \geq 15$.

*(4 marks)*

4. The 9th term of an arithmetic sequence is 37 and the 16th is 65. What is the 20th term?

*(5 marks)*

5. Find the gradient and $y$ – intercept of the lines whose equations are
   (a) $4x - 2y = 10$
   (b) $2x + 3y = 1$
   (c) $y = 4 - x$
   (d) $y = 2x - 5$

*(5 marks)*

6. Twelve workers earn K 225 000 in 6 days. How much will 8 workers earn in 9 days?

*(4 marks)*

7. Express the numbers 1 470 and 7 056 each as a product of its prime factors. Hence evaluate $\frac{1\,470^2}{\sqrt{7\,056}}$, leaving your answer in prime factor form.

*(5 marks)*

8. The interior angle of a regular polygon is 140° more than the exterior

angle. How many sides does the polygon have?

*(4 marks)*

9. Fig. MP 1.1 shows a parallelogram PQRS. Find the sizes of the angles marked *a* , *b* , *c* and *d* .

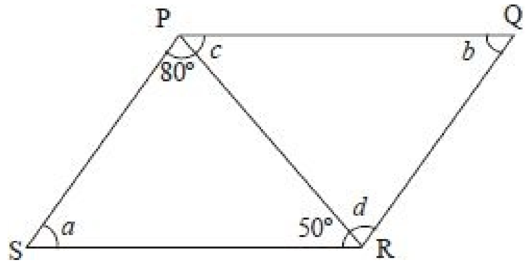



*Fig. MP1.1*

*(7 marks)*

10. Evaluate using logarithms

$$\frac{1.42 - 0.83 \times 0.361}{\sqrt{64} \times 0.0136}$$

*(4 marks)*

11. Work out $\frac{0.37 \times 20.1 - 0.03}{0.08}$

*(4 marks)*

12. The ratio of the lengths of the corresponding sides of two similar rectangular water tanks is 3 : 5. The dimensions of the smaller tank are 1.5 m by 3 m by 4.5 m. Calculate the dimensions of the larger tank.

*(5 marks)*

13. It has been observed that in a certain region, five out of every 100 children die at an age below five years. What is the probability that a child born in the region will live for more than five years?

*(4 marks)*

14. A car dealer charges 2.5% commission for selling a car. After selling the car, he received K 17 500 as commission. How much money did he pass on to the owner of the car from the sale?

*(4 marks)*

15. Given that ΔABC is similar to ΔPQR and that BC = 9.2 cm, CA = 11.4

cm and QR = 13.8 cm, find the ratio of the corresponding sides and hence find the length PR.

*(6 marks)*

16. Using a ruler and pair of compasses only,
    (a) construct triangle PQR with PQ = 2 cm, PR = 5 cm and $\angle QPR = 60°$.
    (b) construct the circumcircle of $\Delta PQR$, measure and state its radius.

*(7 marks)*

17. A cylindrical container of radius 15 cm contains some water. When a solid cube is submerged into the water, the water level rises by 1.2 cm. Find the
    (a) volume of the water displaced.
    (b) the length of the cube.

*(5 marks)*

18. Copy Fig. MP 1.2 and find by construction the centre and angle of rotation that maps $\Delta ABC$ onto $\Delta A'B'C'$.

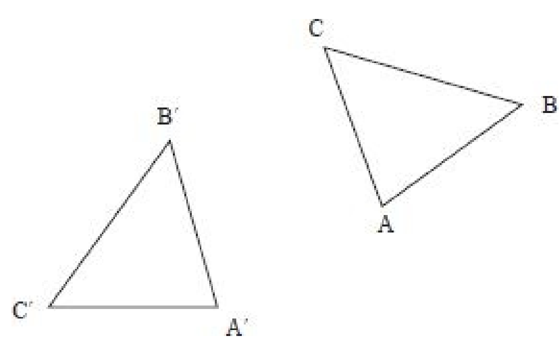



*Fig. MP 1.2*

*(7 marks)*

19. A bus leaves town A at 0900 h and travels at a constant speed of 80 km/h. At 0915 h a car leaves A and travels at a constant speed of 100 km/h. If the distance from A to B is 300 km, draw a distance-time graph showing both journeys. From your graph find when and where the car overtook the bus.

*(7 marks)*

20. Use substitution method to solve the simultaneous equations

$3x - 2y = 10$

$4x + 3y = 2$

*(5 marks)*

# MODEL PAPER II

**Answer all the twenty questions (100 marks).**

1. Solve the simultaneous equations

   $x + y = 7$

   $y = {}^{-}2x + 5$

*(4 marks)*

2. The number of people who attended an agricultural show in one day were 510 men, 1 080 women and some children. When the information was represented on a pie chart, the combined angle for men and children was 216º. Find the angle representing the number of children.

*(4 marks)*

3. Given the numbers 5 and 7, generate a Pythagorean triple.

*(4 marks)*

4. The expression $25x^2 – 80x + k$ is a perfect square. Find the value of $k$ .

*(4 marks)*

5. Work out the following

   (a) $1101_2 + 10101_2$

   (b) $10111_2 – 1111_2$ *(4 marks)*

6. In Fig. MP 2.1, find $x$ .

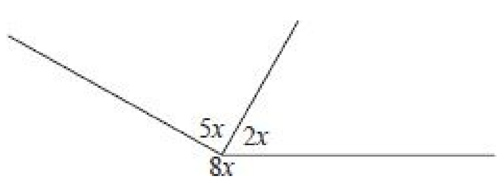



*Fig. MP 2.1*

*(3 marks)*

7. In Fig MP 2.2, find $a, b, c$ .

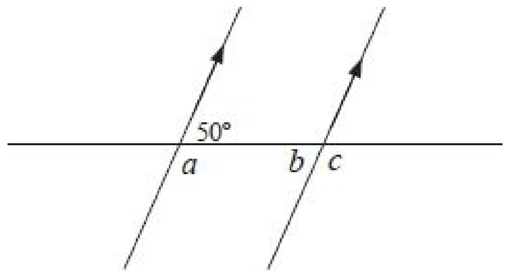



*Fig. MP 2.2*

*(3 marks)*

8. A is fixed point on the circumference of a circle centre O. M is the midpoint of a variable chord AB. Given that O is a fixed point, illustrate and state the locus M.

*(4 marks)*

9. Find the cube root of the following, to 3 s.f.
   (a) 65
   (b) 17.65
   (c) 0.052 8

*(4 marks)*

10. If $\frac{1}{3}\pi r^3 = 25$, find the value of $4\pi r^2$ .

*(4 marks)*

11. How much PAYE is an employee deducted for a monthly pay of K 90 000? Use table MP2.1

*(4 marks)*

| **Taxable income in Kwacha** | | **Rate of tax** |
|---|---|---|
| **MONTHLY** | | |
| 1st | 12 000 | 0% |
| Next | 3 000 | 15% |
| Excess of | 15 000 | 30% |

*Table MP 2.1*

12. A saleslady with an international company based in Blantyre is paid commission as follows:

    12% for sales upto K 120 000 worth of goods, 15% on additional sales between K 120 000 and K 480 000,

18% on additional sales in excess of K 480 000.

If she sold goods worth K 560 000 in a certain month, how much commission did she receive?

*(5 marks)*

13. Generate a sequence given that there are five arithmetic means of terms between 5 and 60.

*(6 marks)*

14. A ladder of 8 m rests against a wall. The foot of the ladder is 2 m from the base of the wall. How high is the point on the wall where the ladder rests?

*(6 marks)*

15. Find the area of the curved surface of a cylinder of radius 7 cm and height 5 cm.

*(4 marks)*

16. An equilateral triangle has sides $x + 4$,
$4x – y$ , $y + 2$. Find its perimeter.

*(4 marks)*

17. Fig. MP 2.3 shows a right pyramid on a rectangular base ABCD, where AB = 14 cm and BC = 10 cm. Its slant edge is 24 cm long. Calculate

    (a) the height of the pyramid.

*(4 marks)*

    (b) the angle between VA and AD.

*(4 marks)*

    (c) Draw the net of the pyramid.

*(4 marks)*

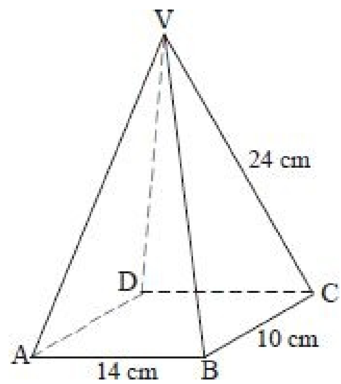



*Fig. MP 2.3*

18. A family consumed 1 220 units of electricity in a month. How much did the family pay given that the fixed charge was K 124.71, the first 30 units are charged at K 2.67 per unit and K 3.91 is charged per unit for the excess of 30 units and less than 750 units, and K 5.55 per unit in excess of 750 units?

*(6 marks)*

19. Construct ΔABC such that,

    (a) AB = 3.5 cm, BC = 6.5 cm and AC = 9 cm

*(3 marks)*

    (b) AB = 6 cm, BC = 7.5 cm and AC = 9 cm

*(3 marks)*

Measure the angles of the triangle in each case.

*(4 marks)*

In Fig. MP 2.4 the rectangles have the same area. What is the difference in their perimeters?

*(5 marks)*

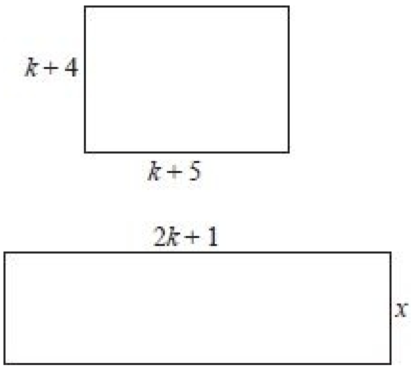



*Fig. MP 2.4*

# LOGARITHMS BASE 10, $x \rightarrow \log_{10} x$

| | | | | | | | | | | | Add Mean Differences | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **1.0** | .0000 | 0043 | 0086 | 0128 | 0170 | 0212 | 0253 | 0294 | 0334 | 0374 | 4 | 8 | 12 | 17 | 21 | 25 | 29 | 33 | 37 |
| **1.1** | .0414 | 0453 | 0492 | 0531 | 0569 | 0607 | 0645 | 0682 | 0719 | 0755 | 4 | 9 | 11 | 15 | 19 | 23 | 27 | 30 | 34 |
| **1.2** | .0792 | 0828 | 0864 | 0899 | 0934 | 0969 | 1004 | 1039 | 1072 | 1106 | 3 | 7 | 10 | 14 | 17 | 21 | 24 | 29 | 31 |
| **1.3** | .1139 | 1173 | 1206 | 1239 | 1271 | 1303 | 1335 | 1367 | 1399 | 1430 | 3 | 6 | 10 | 13 | 16 | 19 | 23 | 26 | 29 |
| **1.4** | .1461 | 1492 | 1523 | 1553 | 1584 | 1614 | 1644 | 1673 | 1703 | 1732 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 |
| **1.5** | .1761 | 1790 | 1818 | 1847 | 1875 | 1903 | 1931 | 1959 | 1987 | 2014 | 3 | 6 | 8 | 11 | 14 | 17 | 20 | 22 | 25 |
| **1.6** | .2041 | 2068 | 2095 | 2122 | 2148 | 2175 | 2201 | 2227 | 2253 | 2279 | 3 | 5 | 8 | 11 | 13 | 16 | 18 | 21 | 24 |
| **1.7** | .2304 | 2330 | 2355 | 2380 | 2405 | 2430 | 2455 | 2480 | 2504 | 2529 | 2 | 5 | 7 | 10 | 12 | 15 | 17 | 20 | 22 |
| **1.8** | .2553 | 2577 | 2601 | 2625 | 2648 | 2672 | 2695 | 2718 | 2742 | 2765 | 2 | 5 | 7 | 9 | 12 | 14 | 16 | 19 | 21 |
| **1.9** | .2788 | 2810 | 2833 | 2856 | 2878 | 2900 | 2923 | 2945 | 2967 | 2989 | 2 | 4 | 7 | 9 | 11 | 13 | 16 | 18 | 20 |
| **2.0** | .3010 | 3032 | 3054 | 3075 | 3096 | 3118 | 3139 | 3160 | 3181 | 3201 | 2 | 4 | 6 | 8 | 11 | 13 | 15 | 17 | 19 |
| **2.1** | .3222 | 3243 | 3263 | 3284 | 3304 | 3324 | 3345 | 3365 | 3385 | 3404 | 2 | 4 | 6 | 9 | 10 | 12 | 14 | 16 | 18 |
| **2.2** | .3424 | 3444 | 3464 | 3483 | 3502 | 3522 | 3541 | 3560 | 3579 | 3599 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| **2.3** | .3617 | 3636 | 3655 | 3674 | 3692 | 3711 | 3729 | 3747 | 3766 | 3784 | 2 | 4 | 6 | 7 | 9 | 11 | 13 | 15 | 17 |
| **2.4** | .3802 | 3820 | 3838 | 3856 | 3874 | 3892 | 3909 | 3927 | 3945 | 3962 | 2 | 4 | 5 | 7 | 9 | 11 | 12 | 14 | 16 |
| **2.5** | .3979 | 3997 | 4014 | 4031 | 4048 | 4065 | 4082 | 4099 | 4116 | 4133 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 15 |
| **2.6** | .4150 | 4166 | 4183 | 4200 | 4216 | 4232 | 4249 | 4265 | 4281 | 4298 | 2 | 3 | 5 | 7 | 8 | 10 | 11 | 13 | 15 |
| **2.7** | .4314 | 4330 | 4346 | 4362 | 4378 | 4393 | 4409 | 4425 | 4440 | 4456 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 13 | 14 |
| **2.8** | .4472 | 4487 | 4502 | 4518 | 4533 | 4548 | 4564 | 4579 | 4594 | 4609 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| **2.9** | .4624 | 4639 | 4654 | 4669 | 4683 | 4698 | 4713 | 4728 | 4742 | 4757 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| **3.0** | .4771 | 4786 | 4800 | 4814 | 4829 | 4843 | 4857 | 4871 | 4886 | 4900 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| **3.1** | .4914 | 4928 | 4942 | 4955 | 4969 | 4983 | 4997 | 5011 | 5024 | 5038 | 1 | 3 | 4 | 6 | 7 | 8 | 10 | 11 | 12 |
| **3.2** | .5051 | 5065 | 5079 | 5092 | 5105 | 5119 | 5132 | 5145 | 5159 | 5172 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 | 12 |
| **3.3** | .5185 | 5198 | 5211 | 5224 | 5237 | 5250 | 5263 | 5276 | 5289 | 5302 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| **3.4** | .5315 | 5328 | 5340 | 5353 | 5366 | 5378 | 5391 | 5403 | 5416 | 5428 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 11 |
| **3.5** | .5441 | 5453 | 5465 | 5478 | 5490 | 5502 | 5514 | 5527 | 5539 | 5551 | 1 | 2 | 4 | 5 | 6 | 7 | 9 | 10 | 11 |
| **3.6** | .5563 | 5575 | 5587 | 5599 | 5611 | 5623 | 5635 | 5647 | 5658 | 5670 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| **3.7** | .5682 | 5694 | 5705 | 5717 | 5729 | 5740 | 5752 | 5763 | 5775 | 5786 | 1 | 2 | 3 | 5 | 6 | 7 | 8 | 9 | 10 |
| **3.8** | .5798 | 5809 | 5821 | 5832 | 5843 | 5855 | 5866 | 5877 | 5888 | 5899 | 1 | 2 | 3 | 5 | 6 | 7 | 9 | 9 | 10 |
| **3.9** | .5911 | 5922 | 5933 | 5944 | 5955 | 5966 | 5977 | 5988 | 5999 | 6010 | 1 | 2 | 3 | 4 | 6 | 7 | 9 | 9 | 10 |
| **4.0** | .6021 | 6031 | 6042 | 6053 | 6064 | 6075 | 6085 | 6096 | 6107 | 6117 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | 10 |
| **4.1** | .6128 | 6138 | 6149 | 6160 | 6170 | 6180 | 6191 | 6201 | 6212 | 6222 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **4.2** | .6232 | 6243 | 6253 | 6263 | 6274 | 6284 | 6294 | 6304 | 6314 | 6325 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **4.3** | .6335 | 6345 | 6355 | 6365 | 6375 | 6385 | 6395 | 6405 | 6415 | 6425 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **4.4** | .6435 | 6444 | 6454 | 6464 | 6474 | 6484 | 6493 | 6503 | 6513 | 6522 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **4.5** | .6532 | 6542 | 6551 | 6561 | 6571 | 6580 | 6590 | 6599 | 6609 | 6618 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **4.6** | .6628 | 6637 | 6646 | 6656 | 6665 | 6675 | 6684 | 6693 | 6702 | 6712 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 7 | 9 |
| **4.7** | .6721 | 6730 | 6739 | 6749 | 6758 | 6767 | 6776 | 6785 | 6794 | 6803 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| **4.8** | .6812 | 6821 | 6830 | 6839 | 6948 | 6857 | 6866 | 6875 | 6884 | 6893 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| **4.9** | .6902 | 6911 | 6920 | 6928 | 6937 | 6946 | 6955 | 6964 | 6972 | 6981 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| **5.0** | .6990 | 6998 | 7007 | 7016 | 7024 | 7033 | 7042 | 7050 | 7059 | 7067 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| **5.1** | .7076 | 7084 | 7093 | 7101 | 7110 | 7118 | 7126 | 7135 | 7143 | 7152 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| **5.2** | .7160 | 7168 | 7177 | 7185 | 7193 | 7202 | 7210 | 7218 | 7226 | 7235 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| **5.3** | .7243 | 7251 | 7259 | 7267 | 7275 | 7284 | 7292 | 7300 | 7308 | 7316 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| **5.4** | .7324 | 7332 | 7340 | 7348 | 7356 | 7364 | 7372 | 7380 | 7388 | 7396 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 1 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

LOGARITHMS BASE 10, $x \rightarrow \log_{10} x$

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Add Mean Differences | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 5.5 | .7404 | 7412 | 7419 | 7427 | 7435 | 7443 | 7451 | 7459 | 7466 | 7474 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 5.6 | .7482 | 7490 | 7497 | 7505 | 7513 | 7520 | 7528 | 7536 | 7543 | 7551 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 5.7 | .7559 | 7566 | 7574 | 7582 | 7589 | 7597 | 7604 | 7612 | 7619 | 7627 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| 5.8 | .7634 | 7642 | 7649 | 7657 | 7664 | 7672 | 7679 | 7686 | 7694 | 7701 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 5.9 | .7709 | 7716 | 7723 | 7731 | 7738 | 7745 | 7752 | 7760 | 7767 | 7774 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| 6.0 | .7782 | 7789 | 7796 | 7803 | 7810 | 7818 | 7825 | 7832 | 7839 | 7846 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 6.1 | .7853 | 7860 | 7868 | 7875 | 7882 | 7889 | 7896 | 7903 | 7910 | 7917 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| 6.2 | .7924 | 7931 | 7938 | 7945 | 7952 | 7959 | 7966 | 7973 | 7980 | 7987 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 6 |
| 6.3 | .7993 | 8000 | 8007 | 8014 | 8021 | 8028 | 8035 | 8041 | 8048 | 8055 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 6.4 | .8062 | 8069 | 8075 | 8082 | 8089 | 8096 | 8102 | 8109 | 8116 | 8122 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 6.5 | .8129 | 8136 | 8142 | 8149 | 8156 | 8162 | 8169 | 8176 | 8182 | 8189 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 6.6 | .8195 | 8202 | 8209 | 8215 | 8222 | 8229 | 8235 | 8241 | 8248 | 8254 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 6.7 | .8261 | 8267 | 8274 | 8280 | 8287 | 8293 | 8299 | 8306 | 8312 | 9319 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| 6.8 | .8325 | 8331 | 8338 | 8344 | 8351 | 8357 | 8363 | 8370 | 8376 | 8382 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 6.9 | .8388 | 8395 | 8401 | 8407 | 8414 | 8420 | 8426 | 8432 | 8439 | 8445 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| 7.0 | .8451 | 8457 | 8463 | 8470 | 8476 | 8482 | 8488 | 8494 | 8500 | 8506 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| 7.1 | .8513 | 8519 | 8525 | 8531 | 8537 | 8543 | 8549 | 8555 | 8561 | 8567 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 7.2 | .8573 | 8579 | 8585 | 8591 | 8597 | 8603 | 8609 | 8615 | 8621 | 8627 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 7.3 | .8633 | 8639 | 8645 | 8651 | 8657 | 8663 | 8669 | 8675 | 8681 | 8686 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 7.4 | .8692 | 8698 | 8704 | 8710 | 8716 | 8722 | 8727 | 8733 | 8739 | 8745 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| 7.5 | .8751 | 8756 | 8762 | 8768 | 8774 | 8779 | 8785 | 8791 | 8797 | 8802 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 7.6 | .8808 | 8814 | 8820 | 8825 | 8831 | 8837 | 8842 | 8848 | 8854 | 8859 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| 7.7 | .8865 | 8871 | 8876 | 8882 | 8887 | 8893 | 8899 | 8904 | 8910 | 8915 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 7.8 | .8921 | 8927 | 8932 | 8938 | 8943 | 8949 | 8954 | 8960 | 8965 | 8971 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 7.9 | .8976 | 8982 | 8987 | 8993 | 8998 | 9004 | 9009 | 9015 | 9020 | 9025 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.0 | .9031 | 9036 | 9042 | 9047 | 9053 | 9058 | 9063 | 9069 | 9074 | 9079 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.1 | .9085 | 9090 | 9096 | 9101 | 9106 | 9112 | 9117 | 9122 | 9128 | 9133 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.2 | .9138 | 9143 | 9149 | 9154 | 9159 | 9165 | 9170 | 9175 | 9180 | 9186 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.3 | .9191 | 9196 | 9201 | 9206 | 9212 | 9217 | 9222 | 9227 | 9232 | 9238 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.4 | .9243 | 9248 | 9253 | 9258 | 9263 | 9269 | 9274 | 9279 | 9294 | 9289 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.5 | .9294 | 9299 | 9304 | 9309 | 9315 | 9320 | 9325 | 9330 | 9335 | 9340 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.6 | .9345 | 9350 | 9355 | 9360 | 9365 | 9370 | 9375 | 9380 | 9385 | 9390 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| 8.7 | .9395 | 9400 | 9405 | 9410 | 9415 | 9420 | 9425 | 9430 | 9435 | 9440 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 8.8 | .9445 | 9450 | 9455 | 9460 | 9465 | 9469 | 9474 | 9479 | 9484 | 9489 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 8.9 | .9494 | 9499 | 9504 | 9509 | 9513 | 9518 | 9523 | 9528 | 9533 | 9538 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.0 | .9542 | 9547 | 9552 | 9557 | 9562 | 9566 | 9571 | 9576 | 9581 | 9586 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.1 | .9590 | 9595 | 9600 | 9605 | 9609 | 9614 | 9619 | 9624 | 9628 | 9633 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.2 | .9638 | 9643 | 9647 | 9652 | 9657 | 9661 | 9666 | 9671 | 9675 | 9680 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.3 | .9685 | 9689 | 9694 | 9699 | 9703 | 9708 | 9713 | 9717 | 9722 | 9727 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.4 | .9731 | 9736 | 9741 | 9745 | 9750 | 9754 | 9759 | 9763 | 9768 | 9773 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.5 | .9777 | 9782 | 9786 | 9791 | 9795 | 9800 | 9805 | 9809 | 9814 | 9818 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.6 | .9823 | 9827 | 9832 | 9836 | 9941 | 9845 | 9850 | 9854 | 9859 | 9863 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.7 | .9868 | 9872 | 9877 | 9881 | 9886 | 9890 | 9894 | 9899 | 9903 | 9908 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.8 | .9912 | 9917 | 9921 | 9926 | 9930 | 9934 | 9939 | 9943 | 9948 | 9952 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| 9.9 | .9956 | 9961 | 9965 | 9969 | 9974 | 9978 | 9983 | 9987 | 9991 | 9996 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 3 | 4 |
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 1 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

**ANTILOGARITHMS $x \rightarrow 10^x$**

| | | | | | | | | | | | Add Mean Differences | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| .00 | 1000 | 1002 | 1005 | 1007 | 1009 | 1012 | 1014 | 1016 | 1019 | 1021 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .01 | 1023 | 1026 | 1028 | 1030 | 1033 | 1035 | 1038 | 1040 | 1042 | 1045 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .02 | 1047 | 1050 | 1052 | 1054 | 1057 | 1059 | 1062 | 1064 | 1067 | 1069 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .03 | 1072 | 1074 | 1076 | 1079 | 1081 | 1084 | 1086 | 1089 | 1091 | 1094 | 0 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 |
| .04 | 1096 | 1099 | 1102 | 1104 | 1107 | 1109 | 1112 | 1114 | 1117 | 1119 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .05 | 1122 | 1125 | 1127 | 1130 | 1132 | 1135 | 1138 | 1140 | 1143 | 1146 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .06 | 1148 | 1151 | 1153 | 1156 | 1159 | 1161 | 1164 | 1167 | 1169 | 1172 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .07 | 1175 | 1179 | 1180 | 1183 | 1186 | 1189 | 1191 | 1194 | 1197 | 1199 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| .08 | 1202 | 1205 | 1208 | 1211 | 1213 | 1216 | 1219 | 1222 | 1225 | 1227 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| .09 | 1230 | 1233 | 1236 | 1239 | 1242 | 1245 | 1247 | 1250 | 1253 | 1256 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| .10 | 1259 | 1262 | 1265 | 1268 | 1271 | 1274 | 1276 | 1279 | 1282 | 1285 | 0 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| .11 | 1288 | 1291 | 1294 | 1297 | 1300 | 1303 | 1306 | 1309 | 1312 | 1315 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| .12 | 1318 | 1321 | 1324 | 1327 | 1330 | 1334 | 1337 | 1340 | 1343 | 1346 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| .13 | 1349 | 1352 | 1355 | 1358 | 1361 | 1365 | 1368 | 1371 | 1374 | 1377 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| .14 | 1380 | 1384 | 1387 | 1390 | 1393 | 1396 | 1400 | 1403 | 1406 | 1409 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 |
| .13 | 1413 | 1416 | 1419 | 1422 | 1426 | 1429 | 1432 | 1435 | 1439 | 1442 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .16 | 1445 | 1449 | 1452 | 1455 | 1459 | 1462 | 1466 | 1469 | 1472 | 1476 | 0 | 1 | 1 | 1 | 2 | 2 | 2 | 3 | 3 |
| .17 | 1479 | 1493 | 1486 | 1489 | 1493 | 1496 | 1500 | 1503 | 1507 | 1510 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .19 | 1514 | 1517 | 1521 | 1524 | 1528 | 1531 | 1535 | 1538 | 1542 | 1545 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .19 | 1549 | 1552 | 1556 | 1560 | 1563 | 1567 | 1570 | 1574 | 1578 | 1581 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .20 | 1585 | 1589 | 1592 | 1596 | 1600 | 1603 | 1607 | 1611 | 1614 | 1618 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .21 | 1622 | 1626 | 1629 | 1633 | 1637 | 1641 | 1644 | 1648 | 1652 | 1656 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .22 | 1660 | 1663 | 1667 | 1671 | 1675 | 1679 | 1683 | 1697 | 1690 | 1694 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .23 | 1698 | 1702 | 1706 | 1710 | 1714 | 1718 | 1722 | 1726 | 1730 | 1734 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .24 | 1738 | 1742 | 1746 | 1750 | 1754 | 1758 | 1762 | 1766 | 1770 | 1774 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .25 | 1778 | 1782 | 1786 | 1791 | 1795 | 1799 | 1803 | 1807 | 1811 | 1816 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .26 | 1820 | 1824 | 1828 | 1832 | 1837 | 1841 | 1845 | 1849 | 1854 | 1858 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .27 | 1862 | 1866 | 1871 | 1875 | 1879 | 1884 | 1888 | 1892 | 1897 | 1901 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .28 | 1905 | 1910 | 1914 | 1919 | 1923 | 1928 | 1932 | 1936 | 1941 | 1945 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .29 | 1950 | 1954 | 1959 | 1963 | 1968 | 1972 | 1977 | 1982 | 1986 | 1991 | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 4 |
| .30 | 1995 | 2000 | 2004 | 2009 | 2014 | 2018 | 2023 | 2028 | 2032 | 2037 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .31 | 2042 | 2046 | 2051 | 2056 | 2061 | 2065 | 2070 | 2075 | 2080 | 2084 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .32 | 2089 | 2094 | 2099 | 2104 | 2109 | 2113 | 2118 | 2123 | 2128 | 2133 | 0 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .33 | 2138 | 2143 | 2148 | 2153 | 2158 | 2163 | 2168 | 2173 | 2178 | 2183 | 1 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 |
| .34 | 2198 | 2193 | 2198 | 2203 | 2208 | 2213 | 2218 | 2223 | 2228 | 2234 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .35 | 2239 | 2244 | 2249 | 2254 | 2259 | 2265 | 2270 | 2275 | 2280 | 2296 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .36 | 2291 | 2296 | 2301 | 2307 | 2312 | 2317 | 2323 | 2328 | 2333 | 2339 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .37 | 2344 | 2350 | 2355 | 2360 | 2366 | 2371 | 2377 | 2382 | 2388 | 2393 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .38 | 2399 | 2404 | 2410 | 2415 | 2421 | 2427 | 2432 | 2438 | 2443 | 2449 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 4 | 5 |
| .39 | 2455 | 2460 | 2466 | 2472 | 2477 | 2483 | 2489 | 2495 | 2500 | 2506 | 1 | 1 | 2 | 2 | 3 | 3 | 4 | 5 | 5 |
| .40 | 2512 | 2518 | 2523 | 2529 | 2535 | 2541 | 2547 | 2553 | 2559 | 2564 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| .41 | 2570 | 2576 | 2582 | 2588 | 2594 | 2600 | 2606 | 2612 | 2618 | 2624 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 5 |
| .42 | 2630 | 2636 | 2642 | 2649 | 2655 | 2661 | 2667 | 2673 | 2679 | 2685 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 5 | 6 |
| .43 | 2692 | 2698 | 2704 | 2710 | 2716 | 2723 | 2729 | 2735 | 2742 | 2748 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| .44 | 2754 | 2761 | 2767 | 2773 | 2780 | 2786 | 2793 | 2799 | 2805 | 2812 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 5 | 6 |
| .45 | 2818 | 2825 | 2831 | 2838 | 2844 | 2851 | 2858 | 2864 | 2871 | 2877 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| .46 | 2884 | 2891 | 2897 | 2904 | 2911 | 2917 | 2924 | 2931 | 2938 | 2944 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| .47 | 2951 | 2958 | 2965 | 2972 | 2979 | 2985 | 2992 | 2999 | 3006 | 3013 | 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 |
| .48 | 3020 | 3027 | 3034 | 3041 | 3048 | 3055 | 3062 | 3069 | 3076 | 3083 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| .49 | 3090 | 3097 | 3105 | 3112 | 3119 | 3126 | 3133 | 3141 | 3148 | 3155 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 6 |
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

**ANTILOGARITHMS $x \to 10^x$**

| | | | | | | | | | | | Add Mean Differences | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *x* | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |
| **.50** | 3162 | 3170 | 3177 | 3184 | 3192 | 3199 | 3206 | 3214 | 3221 | 3228 | 1 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 |
| **.51** | 3236 | 3243 | 3251 | 3258 | 3266 | 3273 | 3281 | 3289 | 3296 | 3304 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| **.52** | 3311 | 3319 | 3327 | 3334 | 3342 | 3350 | 3357 | 3365 | 3373 | 3381 | 1 | 2 | 2 | 3 | 4 | 5 | 5 | 6 | 7 |
| **.53** | 3388 | 3396 | 3404 | 3412 | 3420 | 3428 | 3436 | 3443 | 3451 | 3459 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| **.54** | 3467 | 3475 | 3483 | 3491 | 3499 | 3508 | 3516 | 3524 | 3532 | 3540 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 6 | 7 |
| **.55** | 3548 | 3556 | 3565 | 3573 | 3581 | 3589 | 3597 | 3606 | 3614 | 3622 | 1 | 2 | 2 | 3 | 4 | 5 | 6 | 7 | 7 |
| **.56** | 3631 | 3639 | 3648 | 3656 | 3664 | 3673 | 3681 | 3690 | 3698 | 3707 | 1 | 2 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |
| **.57** | 3715 | 3724 | 3733 | 3741 | 3750 | 3758 | 3767 | 3776 | 3784 | 3793 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| **.58** | 3802 | 3811 | 3819 | 3828 | 3837 | 3846 | 3855 | 3864 | 3873 | 3882 | 1 | 2 | 3 | 4 | 4 | 5 | 6 | 7 | 8 |
| **.59** | 3890 | 3899 | 3908 | 3917 | 3926 | 3936 | 3945 | 3954 | 3963 | 3972 | 1 | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 8 |
| **.60** | 3981 | 3990 | 3999 | 4009 | 4018 | 4027 | 4036 | 4046 | 4055 | 4064 | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | 8 |
| **.61** | 4074 | 4083 | 4093 | 4102 | 4111 | 4121 | 4130 | 4140 | 4150 | 4159 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **.62** | 4169 | 4178 | 4188 | 4198 | 4207 | 4217 | 4227 | 4236 | 4246 | 4256 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **.63** | 4266 | 4276 | 4285 | 4295 | 4305 | 4315 | 4325 | 4335 | 4345 | 4355 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 9 |
| **.64** | 4365 | 4375 | 4385 | 4395 | 4406 | 4416 | 4426 | 4436 | 4446 | 4457 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 9 |
| **.65** | 4467 | 4477 | 4487 | 4498 | 4508 | 4519 | 4529 | 4539 | 4550 | 4560 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| **.66** | 4571 | 4581 | 4592 | 4603 | 4613 | 4624 | 4634 | 4645 | 4656 | 4667 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| **.67** | 4677 | 4688 | 4699 | 4710 | 4721 | 4732 | 4742 | 4753 | 4764 | 4775 | 1 | 2 | 3 | 4 | 5 | 7 | 8 | 9 | 10 |
| **.68** | 4786 | 4797 | 4808 | 4819 | 4831 | 4842 | 4853 | 4864 | 4875 | 4887 | 1 | 2 | 3 | 4 | 6 | 7 | 9 | 9 | 10 |
| **.69** | 4898 | 4909 | 4920 | 4932 | 4943 | 4955 | 4966 | 4977 | 4989 | 5000 | 1 | 2 | 3 | 5 | 6 | 7 | 9 | 9 | 10 |
| **.70** | 5012 | 5023 | 5035 | 5047 | 5058 | 5070 | 5082 | 5093 | 5105 | 5117 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| **.71** | 5129 | 5140 | 5152 | 5164 | 5176 | 5188 | 5200 | 5212 | 5224 | 5236 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| **.72** | 5248 | 5260 | 5272 | 5284 | 5297 | 5309 | 5321 | 5333 | 5346 | 5358 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| **.73** | 5370 | 5383 | 5395 | 5408 | 5420 | 5433 | 5445 | 5458 | 5470 | 5483 | 1 | 2 | 4 | 5 | 6 | 7 | 8 | 10 | 11 |
| **.74** | 5495 | 5508 | 5521 | 5534 | 5546 | 5559 | 5572 | 5595 | 5598 | 5610 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 10 | 12 |
| **.75** | 5623 | 5636 | 5649 | 5662 | 5675 | 5689 | 5702 | 5715 | 5728 | 5741 | 1 | 3 | 4 | 5 | 7 | 8 | 9 | 10 | 12 |
| **.76** | 5754 | 5768 | 5781 | 5794 | 5808 | 5821 | 5834 | 5948 | 5961 | 5875 | 1 | 3 | 4 | 5 | 7 | 9 | 9 | 11 | 12 |
| **.77** | 5998 | 5902 | 5916 | 5929 | 5943 | 5957 | 5970 | 5994 | 5998 | 6012 | 1 | 3 | 4 | 5 | 7 | 9 | 10 | 11 | 12 |
| **.75** | 6026 | 6039 | 6053 | 6067 | 6091 | 6095 | 6109 | 6124 | 6138 | 6152 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| **.79** | 6166 | 6190 | 6194 | 6209 | 6223 | 6237 | 6252 | 6266 | 6281 | 6295 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 11 | 13 |
| **.80** | 6310 | 6324 | 6339 | 6353 | 6369 | 6383 | 6387 | 6412 | 6427 | 6442 | 1 | 3 | 4 | 6 | 7 | 9 | 10 | 12 | 13 |
| **.81** | 6457 | 6471 | 6486 | 6501 | 6516 | 6531 | 6546 | 6561 | 6577 | 6592 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| **.82** | 6607 | 6622 | 6637 | 6653 | 6668 | 6683 | 6699 | 6714 | 6730 | 6745 | 2 | 3 | 5 | 6 | 8 | 9 | 11 | 12 | 14 |
| **.83** | 6761 | 6776 | 6792 | 6808 | 6823 | 6839 | 6855 | 6871 | 6887 | 6902 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 14 |
| **.84** | 6918 | 6934 | 6950 | 6966 | 6982 | 6998 | 7015 | 7031 | 7047 | 7063 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 14 |
| **.85** | 7079 | 7096 | 7112 | 7129 | 7145 | 7161 | 7178 | 7194 | 7211 | 7229 | 2 | 3 | 5 | 6 | 8 | 10 | 11 | 13 | 14 |
| **.86** | 7244 | 7261 | 7278 | 7295 | 7311 | 7328 | 7345 | 7362 | 7379 | 7396 | 2 | 3 | 5 | 7 | 8 | 10 | 12 | 14 | 15 |
| **.87** | 7413 | 7430 | 7447 | 7464 | 7482 | 7499 | 7516 | 7534 | 7551 | 7568 | 2 | 3 | 5 | 7 | 9 | 10 | 12 | 14 | 16 |
| **.88** | 7586 | 7603 | 7621 | 7638 | 7656 | 7674 | 7691 | 7709 | 7727 | 7745 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| **.89** | 7762 | 7780 | 7798 | 7816 | 7834 | 7852 | 7870 | 7889 | 7907 | 7925 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| **.90** | 7943 | 7962 | 7980 | 7998 | 8017 | 8035 | 8054 | 8072 | 8091 | 8110 | 2 | 4 | 5 | 7 | 9 | 11 | 13 | 14 | 16 |
| **.91** | 8128 | 8147 | 8166 | 8185 | 8204 | 8222 | 8241 | 8260 | 8279 | 8299 | 2 | 4 | 6 | 8 | 10 | 11 | 13 | 15 | 17 |
| **.92** | 8318 | 9337 | 8356 | 8375 | 8395 | 8414 | 8433 | 8453 | 8472 | 8492 | 2 | 4 | 6 | 8 | 10 | 11 | 13 | 15 | 17 |
| **.93** | 8511 | 8531 | 8551 | 8570 | 8590 | 8610 | 8630 | 8650 | 8670 | 8690 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| **.94** | 8710 | 8730 | 8750 | 8770 | 8790 | 8810 | 8831 | 8851 | 8872 | 8892 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
| **.95** | 8913 | 8933 | 8954 | 8974 | 8995 | 9016 | 9036 | 9057 | 9078 | 9099 | 2 | 4 | 6 | 8 | 10 | 13 | 15 | 17 | 19 |
| **.96** | 9120 | 9141 | 9162 | 9183 | 9204 | 9226 | 9247 | 9268 | 9290 | 9311 | 2 | 4 | 6 | 9 | 11 | 13 | 15 | 17 | 19 |
| **.97** | 9333 | 9354 | 9376 | 9397 | 9419 | 9441 | 9462 | 9484 | 9506 | 9528 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 18 | 20 |
| **.98** | 9550 | 9572 | 9594 | 9616 | 9638 | 9661 | 9683 | 9705 | 9727 | 9750 | 2 | 4 | 7 | 9 | 11 | 13 | 15 | 18 | 20 |
| **.99** | 9772 | 9795 | 9817 | 9840 | 9863 | 9886 | 9908 | 9931 | 9954 | 9977 | 2 | 5 | 7 | 9 | 11 | 14 | 16 | 18 | 21 |
| *x* | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** |

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